Locality and causality properties of light cone string field theory

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A Thesis presented for the degree of
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Dedicated to my family

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Abstract

The recent discovery of a new maximally supersymmetric background for the type IIB superstring theory has revived the interest in light cone string field theory. This background is a plane wave background with the additional support of a non-trivial self dual Ramond-Ramond 5-form field strength. It can be quantised in the light cone gauge and hence it naturally fits into the framework of light cone string field theory.

In this thesis we re-examine the causality and locality properties of string theory in the flat background and compare it with the recent results for string theory in this plane wave background. We formulate the causality requirement in terms of the commutativity of the string field, as it is usually done in point particle field theory.

We find that the string light cone in the plane wave background shares similar properties with the string light cone in the flat background. Even more interesting is that, unlike the flat background theory, string interactions in the plane wave background do not modify the causal structure of the theory. This has interesting consequences for the choice of the 3-string vertex in the plane wave background, a topic that is still an active subject of research.

Declaration

The work in this thesis is based on research carried out at the Center for Particle Theory, Department of Mathematical Sciences, University of Durham, England. No part of this thesis has been submitted elsewhere for any other degree or qualification and it all my own work unless referenced to the contrary in the text. Section (4.3.2) of chapter 4 is based on joined research with my supervisor Dr. Chong-Sun Chu, published on [1]. Section (5.2) of chapter 5 is based on joined research with my supervisor Dr. Chong-Sun Chu, published in [2]. Section C.2 is based on joined research with my supervisor Dr. Chong-Sun Chu (unpublished). References to other people's work are given as appropriate throughout the text.

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Chapter 1

Introduction

Causality means that the cause must precede the effect. Two events are causally related if the past event could in principle influence the future event. Since every physical process is at the bottom line a scattering process, we can say that two events are causally related if an object could travel from the past event to the future event. In order to determine the causal relation of two events, it is not necessary for such an object to actually be involved, we are not studying the details of the process. It is only sufficient to determined if it could exist in principle. If the answer to this question is "yes", then the two events are causally related, otherwise they are not.

1.1 Causality in Classical Physics

In Newtonian physics, the answer to this question is always "yes". There is absolutely no reason why any physical object can not travel fast enough and thus starting from one event, to reach any future event. However we know that Newtonian physics are only an approximation. A more rigorous theory (that has past so far every experimental test) is special relativity[1]. In special relativity, the notion of causality changes radically.

[1] A good book dedicated to special relativity is [3]. Alternatively, the first chapter of books on general relativity provide a good review of special relativity. A few that might be of interest are [4], [5], [6], [7].

Special Relativity places an upper bound on the speed an object can achieve. This is of course arithmetically equal to the speed of light in the vacuum c. Nothing with mass can travel faster than light and only massless particles can travel exactly with the speed of light. Therefore, it must be clear, even to the non expert, that now not all events can be causally related to each other. Two events, that their causal connection would require a mediator travelling at superluminal speed can not be causally related any more.

Given an event (chosen arbitrarily), we can construct a hypersurface in spacetime that will divide spacetime down to two regions. One will contain all the events that are causally related to our reference event (past and future) and a second region of all the events that are causally unrelated. This hypersurface has a conical shape and it is called the *light cone*. Events falling inside the light cone are the causally related events and we say that they are *timelike* separated, while events that fall outside the light cone are those that are causally unrelated and we say that they are *spacelike* separated. The marginal case, namely events that fall exactly on the light cone, are considered to be causally related, only that it requires for the mediator object to travel exactly at the speed of light. These events are called *lightlike* separated.

1.2 Causality in Quantum Physics

This however is not the end of our story. We know that at a very fundamental level the Universe is quantum mechanical. Therefore, we have to examine causality within the framework of quantum mechanics. Naively, we would ask whether a particle can propagate from one event to an other. We would expect for the amplitude of propagation to be zero outside the light cone, but it turns out that it is not, even when we are working with the relativistic energy–momentum relation.

But we are forgetting something here. Quantum mechanical particles are not like classical ones, with well defined trajectories, momenta and energies. In fact, the Heisenberg uncertainty principle permits them to do all sorts of unnatural things, as long as they can go undetected. The key word here is detection (or measurement). Causality in a quantum theory will be preserved if measurements at spacelike separations are not allowed to interfere. Physical quantities that we can measure correspond to (hermitian) operators. The condition that the measurements of a physical quantity at two different spacetime points, which are spacelike separated, will not interfere, translates into the requirement that the commutator of the corresponding operator vanishes identically.

The marriage of special relativity and quantum mechanics has been possible under the roof of quantum field theory[2]. It is within this formalism that we should examine causality. The basic object in quantum field theory is the field, a function over spacetime that we promote to an operator. The field can or can not be an observable by itself. But every observable of the theory is built up by the field and its derivatives. Therefore, causality in quantum field theory is formulated as the local commutativity of the fields. The requirement for causality translates into the requirement that the commutator of the field vanishes identically for spacelike separations. Then it can been shown that quantum field theory does indeed preserve causality as we know it from classical physics.

This idea of causality is called *microcausality* or *microscopic causality*. This is because we require the commutator of the field to vanish for spacelike separations, no matter how closed they are spatially. We make this distinction because in a quantum theory one can examine causality using the S-matrix. The S-matrix is defined as the operator that gives the amplitude of transition of a system, from a known state in the far past to another state in the far future. In a relativistic theory, states have specific Lorentz transformation properties. Lorentz invariance of the theory implies the Lorentz transformation of the S-matrix. If we can construct such a theory, it would preserve causality. It turns out however, that this can be achieved if the theory is formulated in terms of fields, if it is a quantum field theory. Then, in order for the S-matrix to have the correct Lorentz transformation properties, we need the commutator of the fields to vanish outside the light cone and the interactions to be local in the fields. The later condition is the requirement of locality.

Causality studied with S-matrix would correspond to macroscopic causality in

[2]For an introduction to quantum field theory see [8], while [9], [10] and [11] offer a more thorough treatment.

classical terms. But the conditions of locality and microcausality (formulated as the commutability of the fields) ensure macroscopic causality. Although one can go back and forth between the two formulations of causality, it is our feeling that microcausality, with the addition of locality, is more fundamental than macroscopic causality.

1.3 String Theory

It would be natural to ask the same questions for string theory, namely if the theory is causal and how. But first we have to see what is string theory.

1.3.1 String theory in a nutshell

As the name suggests, it is a theory studying the motion of a string. A string in return is an one-dimensional extended object. It can be open or closed.

String theory[3] aspires to be a unified theory of everything. As far as our knowledge goes, there are four interactions in nature (electromagnetic, weak nuclear, strong nuclear and gravitational). The first three are described by a quantum field theory, to a very high accuracy. Electromagnetic and weak interactions are unified under the umbrella of electroweak theory. Gravity on the other hand seems to escape a consistent quantisation. Even with the inclusion of supersymmetry, a consistent quantum theory seems to be elusive, the problem being that the theory is non-renormalizable.

String theory promises to bring all four interactions and all the elementary particles of nature under the same roof. This is done as follows. In ordinary field theory (like the Standard Model), every particle is perceived as a fundamentally different entity. It is for this reason that each particle species is assigned a different field. On the other hand, in string theory there can be two kinds of strings at most, open and closed strings.

[3]String theory is well covered in the literature, with many books available. The reader may consult wish to consult any of [12], [13], [14], [15], [16], [17], [18], [19], [20], [21] for a review of string theory.

However, there is a major difference between strings and point particles. Point particles can have one kind of motion, that is, the only thing they can do is move around, from here to there. Strings on the other hand have two kinds of motion. First, they can move around as whole objects. This is no different from what point particles can do. But there is a second kind of motion that they can perform and that is that they can oscillate. This is in a sense an internal motion of the string, something that the point particle lacks.

The string can not oscillate in any way it wishes, but only at specific ways, called modes of oscillation. These modes turn out to behave like different particles. Strings vibrating in a certain way behave like scalar particles, strings vibrating in a different way as gauge bosons and, with the inclusion of supersymmetry, we can have string vibrating in such a way as to be spin one-half fermions. Instead of having many different elementary particles as the building blocks of the Cosmos, one has only two, an open string and a closed string. It is because of their nature as extended objects and their modes of internal oscillations that we can perceive them as different objects. Let us give an example from everyday life . Our friend George can appear different from day to day because of the way he dresses. But he is always the same person. The same applies here. A string can appear as a different particle, but it is always the same object.

String theory was first brought into physics in the late 60's[4] to explain the high number of strongly interacting particles produced in the experiments. Although it was a promising theory, it suffered from two major drawbacks. One was that consistency of the quantum theory demanded that spacetime had twenty six dimensions instead of the usual four. Even with the addition of fermions to the string model, the dimensions of spacetime had to be ten, six more than observed. Furthermore, the closed string sector had naturally a massless spin 2 particle that could not be identified with any particle observed. Add to that the arrival of the quark model and the description of strong interaction by an $SU(3)$ Yang Mills theory and it is no surprise that string theory was abandoned as a theory for strong interactions.

[4]It was the work of Veneziano [22] that introduced the dual models into high energy physics. A little bit later it became clear that these corresponded to strings.

But the spin 2 particle was to become the most prominent feature of string theory. It turned out that this particle had all the properties of the graviton, the hypothetical particle that mediates the gravitational interaction. Closed string theory (and its supersymmetric extension) among other things includes gravity. Starting with a closed string in a Minkowski spacetime, one can get gravitational interactions without imposing any additional requirements. Add to that the fact that the open string naturally includes a vector gauge boson (which would correspond to a Yang Mills theory) and the picture is almost complete. A theory built upon open and closed strings can in principle accommodate all the known interactions of the Standard Model plus gravity. For a theory that would include in the same way fermions and it will also be consistent and without anomalies, one has to include supersymmetry. Even then it turns out that there are only five acceptable theories, which are called Type I, Type IIA & IIB, Heterotic $SO(32)$ and Heterotic $E_8 \times E_8$.

But this is not all in string theory. String theory is more than a theory of strings. It turns out that it contains other dynamical objects of higher dimensionality called D-branes[5]. These objects were originally discovered by studying open strings with Dirichlet boundary conditions (hence the designation "D"). Originally Dirichlet boundary conditions were disregarded because they would break spacetime Lorentz symmetry[6]. But it turned out that they can be imposed consistently, provided one includes an additional object in the theory, the D-brane. These D-branes are solitonic solutions of string theory.

Furthermore, it was discovered that the five superstring theories are not totally independent. There is a web of dualities that can turn one into the other. And what is even more bizarre, in a certain limit type IIA string theory turns out to be the limit of the 11-dimensional theory, which nobody knows the exact context. This theory has been named M-Theory[7] and its low energy limit is the 11-dimensional supergravity. There are some arguments that M-Theory might be a theory of membranes, but so far, to the best of our knowledge this is more a speculation than

[5]For a comprehensive review see [20]

[6]The first time that Dirichlet boundary conditions were considered in string theory was in [23].

[7]See [24], [25]. For a review on M-theory, see [26].

proof.

Although string theory has been studied in the flat Minkowski spacetime, there are other backgrounds that are as interesting, if not more. For string theory one such background is the $AdS_5 \times S^5$, which is a solution of the type IIB supergravity. String theory in this background is hard to solve and in practise we have been unable to go beyond the supergravity limit. Nevertheless, a very interesting feature arose. String theory in this background is believed to be dual to a super Yang Mills with $\mathcal{N} = 4$ supersymmetry, living on the boundary. This is the celebrated AdS/CFT correspondence[8]. Dual theories means that we can calculate propagators and amplitudes of one theory using the other theory. The duality is a strong/weak coupling duality, which means that a calculation in one theory for coupling constant g corresponds to a calculation in the dual theory for coupling constant $g' \sim 1/g$. This in return gave rise to the idea of holography[9], which states that physics in the bulk can equally well be described by the physics on the boundary. This comes in touch with black hole[10] thermodynamics, where the entropy of the black hole depends on the area of the event horizon, not on the volume[11].

String theory has been well developed as a first quantised theory. By that we mean that when we quantise the theory we promote to operators the position of the string and the conjugate momentum. Effectively, this means that we study the motion of a single string. This formulation of the theory has been successful, but it has its limitations.

From the point particle theories, we know that a better formulation would be one based on fields[12]. Perturbation theory is easier to derive, symmetries to be incorporated and most of all, it is the natural framework for the study of non-perturbative phenomena. The same construction in string theory has not been

[8]The correspondence originated from the Maldacena conjecture [27]. Some reviews are [28], [29], [30], [31], [32].

[9]See [33], [34] and for reviews [35], [36].

[10]For a review of black holes in string theory, see [37].

[11]The volume of the black hole is defined as the volume enclosed by the event horizon.

[12]According to Weinberg, reconciling special relativity with quantum mechanics leads necessarily to quantum field theory, see [9].

successful yet. We can write down a full string field theory, for open and closed string, both bosonic and supersymmetric in the light cone gauge[13]. The advantage is that the theory is unitary, without ghosts and easy to quantise in a canonical way. The price we pay is that manifest spacetime covariance is lost and so is string gauge symmetry.

For open bosonic string, we do have a covariant and gauge invariant string field theory by Witten [51]. But for closed string and/or superstrings, things are not very clear yet. In particular, the problems encountered in a closed string field theory is that the action seems to be non-polynomial. The extension to superstrings can, in principle, be accomplished in two ways. One is to base the analysis on the RNS formalism of the superstring, which as a first quantised theory has been quantised with covariant methods. For open superstrings, one can write a theory very similar to Witten's. The problem here is that on has to include an extra operator in the kinetic term, called "picture changing operator". The other way would be to base the construction on the GS formalism. However, the GS superstring has eluded so far our attempts for a covariant quantisation (although progress has been made with the recent work of Berkovits [52], [53], [54], we feel that there is still work to be done). Until this problem is solved, we can not go far with a full superstring theory based on the GS superstring model. To that, one should add the problems of quantising string field theory. In particular, although there is a quantisation procedure for Witten's theory [55] using the Batalin-Vilkovisky approach (see [56], [57] for a review), we feel that the question on how to incorporate closed strings has not been answered satisfactorily. We know that a string theory based on open string inevitably includes closed strings as well. Without a satisfactory closed string field theory, any open string field theory will be incomplete. For these reason, we will restrict ourselves to the light cone string field theory.

[13]Light cone string field theory originated from the functional methods developed by Mandelstam to describe the 3-string interaction in [38] for the bosonic and in [39] for the Ramond-Neveu-Schwarz. String field theory in the light cone gauge was further developed in [40], [41], with further contributions in [42], [43], [44]. The extension to the superstring, based on the Green-Schwarz model was carried out in [45], [46], [47], [48]. Two very good reviews are [49], [50]

Thus, we see that string theory is more than one would originally guess from the name. But the discussion has taken us far from our original subject, causality.

1.3.2 Causality in String Theory

Strings are extended objects, so it would be interesting (to say the least) to examine causality in the string sense. Again there are two ways to proceed in a quantum theory of strings. The first is to examine what would be microcausality, i.e. the commutability of the string field. The other would be to examine the string S-matrix. Here we will present what would be microscopic causality in string theory. For that purpose we will use string field theory in the light cone gauge.

The study of causality in string theory mimics the study of causality in field theory. One first writes down a string field theory, which for our purpose will be in the light cone gauge. Then we express the requirement of causality as the condition for the commutator of two string fields to vanish identically. The condition that will emerge, will give us the string light cone. The result is quite surprising and different from the point particle case. The string light cone is different form the particle light cone, the modification coming from the internal oscillating modes of the string. However one finds that the zero mode of the string, which corresponds to the centre of mass of the string, behaves like a particle. Furthermore, if one truncates the string field down to component fields, then one recovers the point particle light cone. For string theory in the flat, Minkowski spacetime, this calculation was first done in [58], with further comments in [59], then extended to the superstrings in [60].

However, things become more interesting when one examines how interactions affect the string light cone. For string field theory in a flat, Minkowski, background it was first found in [61] and then in [62] that amplitudes receive contributions outside the string light cone. Interactions do modify the string microcausality.

In that light, it was interesting to consider how things get modified in the plane wave background[14]. Plane waves for string theory arose as Penrose limits of $AdS_5 \times S^5$. String theory (more specifically type IIB superstring theory) in this background

[14]This is also referred in the literature as the pp-wave background.

can be solved exactly in the light cone gauge. Based on that one can write a string field theory in the light cone gauge, with all the advantages and disadvantages of the same theory in the flat background. One important property of the plane wave background is that the metric is dependant on a parameter μ (with units of mass). When one takes the limit $\mu \to 0$, the plane wave background reduces smoothly to the flat one. It is natural that we demand the plane wave string field theory to reduce to the flat string field theory in this limit. But things turn out to be more interesting than this.

Being able to study and solve string theory in a non-trivial background other than the flat one is not a common thing and should not been taken lightly. We are really lucky to be able to do so for a string in a plane wave background and we should take up the opportunity to explore string theory as far as possible.

But the importance of the plane wave does not stop here. As we said, plane waves originated from $AdS_5 \times S^5$. String theory in $AdS_5 \times S^5$ is conjectured to be dual to a $\mathcal{N} = 4$ Super Yang Mills. So far, as we said before, we have not been able to study string theory in the $AdS_5 \times S^5$ beyond the supergravity level. But if we can study string theory in a certain limit of $AdS_5 \times S^5$, namely the plane wave, and solve it, then we could go ahead with the correspondence. In fact it has been proposed that string theory in the plane wave is dual to a certain sector of the super Yang Mills. This is the famous BMN limit and has attracted a lot of attention lately.

Studying the causal structure of string theory in the plane wave is the next step. It has been found in [1] that for the free theory, the string light cone exhibits the same departure from the corresponding point particle light cone, due to the internal oscillating modes of the string. Furthermore, in the $\mu \to 0$ limit, the plane wave string light cone reduces to the flat background string light cone.

The really interesting part is when one wants to study how interactions in the plane wave modify the string light cone. The surprising thing is that they do not! Contrary to the flat case background, in the plane wave background the interaction do not modify the string light cone. This is quite surprising and suggests that at a deepest level string theory in the plane wave background in the limit $\mu \to 0$ is not going smoothly to string theory in the flat background.

1.4 Outline

The structure of the thesis is as follows.

In chapter 2 we provide an overview of light cone string field theory in the flat background. After some preliminaries for the nature of the string field, we formulate string field theory for open bosonic string, free and interacting. Emphasis is given in the 3-string interaction. Then we proceed with formulating the closed string field theory. Finally, we proceed with the supersymmetric extension. Throughout the chapter, we emphasize on the methodology of constructing the string field theory, leaving the details for the literature. Our ultimate purpose is to use string field theory to study the causal structure of string theory and for that purpose we emphasize on those aspects that we will need later on.

In chapter 3 we provide a brief overview of string theory in the plane wave background. After explaining how the plane wave came into string theory and its importance, we demonstrate how the theory can be solved as a first quantised one. Based on that, we proceed with the formulation of a string field theory. Again, our purpose is to give an overview, emphasizing on the key points of the theory that we will need later on, leaving the details for the literature.

In chapter 4 we study (micro)causality in string theory. We start with a more detailed discussion of the light cone in classical physics and causality in quantum field theory, only to proceed and discuss the string light cone in both the flat and the plane wave background for a free string. The construction of the string light cone, as we will see, mimics the construction of the light cone of a point particle using field theory methods. We conclude this chapter with a few important remarks.

Then in chapter 5 we examine how string interactions affect the causal structure of sting theory. There we will see how the string light cone gets modified by interaction in a flat background, while it does not acquire any additional contributions in the plane wave background.

Finally, in chapter 6 we collect the main results of chapters 4 and 5 for a more detailed discussion. In addition we discuss about how one could go beyond light cone string field theory and study causality. We conclude with open questions that we feel should be addressed sooner or later.

In the appendices we have included material that although useful, somehow fall outside the main line of argument. In order not to disrupt the reasoning, we preferred to included them as appendices. Appendix A provides a brief review of the first quantised string in the light cone gauge. Appendix B deals with applications of the causality condition (formulated as the local commutability of the fields) in point particle field theory. Appendix C presents the study of a certain amplitude in both the flat and the plane wave background.

Chapter 2

Light Cone String Field Theory

In this chapter we will provide a brief introduction to the field theory of strings. There is no intention to present a full account of string field theory, that would be impossible in the limited space we have. Rather, we will restrict ourselves on string field theory in the light cone gauge, since this theory will be sufficient for our further discussion.

After a few important preliminaries, we will formulate a string field theory for the open bosonic strings propagating in a flat Minkowski spacetime. In developing the theory we will follow the analogy with developing a field theory for point particles. Then we present the closed strings theory. Finally we proceed with the formulation of the supersymmetric string field theory, still in the light cone gauge.

Throughout this chapter we assume that the reader is familiar with the first quantised theory of strings. The subject is well covered in the literature, see [12], [13], [14], [15], [16], [17], [18], [19], [20], [21]. The light cone gauge was first introduced for strings in [63]. Bosonic string field theory in the light cone gauge for a string propagating in a Minkowski spacetime is discussed in [40], [41], [44]. The 3-string vertex is further analyzed in [38], [43], [42]. Superstring field theory was developed in [45], [46], [48]. There are also two very good reviews of superstring field theory, [50], [49].

We work in units where $\hbar = c = 1$, but we keep α' arbitrary. We leave the dimensionality of spacetime, D, arbitrary with the understanding that $D = 26$ for bosonic strings and $D = 10$ for superstrings.

2.1 Towards a Field Theory of Strings

A point particle field mathematically is a map from the points of a manifold (usually this is the Minkowski spacetime), to a set of numbers. Take for example the simplest case, the real Klein–Gordon field. It is just a function $\phi(x^\mu)$. With that on hand, one seeks an action, written as the integral of a Lagrangian. The equations of motion then, i.e. the differential equations that the field obeys, are easy to derive from the requirement $\delta S = 0$. By solving them, one knows how the field behaves in a given situation. Quantizing, means that the function $\phi(x^\mu)$ is promoted to an operator. Then, one expands in a set of complete eigenstates. For the free theory it turns out that these states describe a finite and discrete sets of particles. Of course this can be generalized to define fields that give complex values or define vector, tensor, spinor fields.

We would like to do the same with strings. The reason is that in the case of point particles this turns out to be the natural framework to describe interactions and systems of many particles (especially when their numbers vary). Furthermore, although mathematical difficulty forces us sometimes to have only perturbative solutions, this is the appropriate framework for studying non-perturbative phenomena, like solitons and symmetry breaking. Even perturbation theory comes out more naturally from a field theory. So far, for strings we have a so called "first quantised theory", i.e. a theory where we are concerned with and study the motion of only one string. It is only natural to try to formulate a field theory for strings.

A string is an one-dimensional extended object, either open (like a piece of rope) or closed (like a loop). Mathematically this is described as a curve. Generalizing the concept of a point particle field, we claim that the string field will be a map from curves of the spacetime manifold to real numbers. Recall that a curve in a manifold can be parameterized with a parameter, say σ. Then every point along the curve is determined by its coordinates, $X^\mu(\sigma)$, functions of the parameter σ. However the concept of the curve is more fundamental than its parametrization. After all, for the same curve (the same geometrical object) we can use many different parameterizations. But the object itself remains the same. Therefore, this map from curves to numbers has to be a functional that we will denote by $\Phi[X(\sigma)]$ for open strings and $\Psi[X(\sigma)]$ for closed strings. There is no explicit dependance on σ. Each curve (string) is mapped to a number and this is independent of the parametrization we chose for the curve.

In the framework of point particle physics, we treat every elementary particle species as a fundamentally different entity and therefore we assign a different field to each of them. An electron is different from a photon and they are both different from the Higgs particle. Their properties dictate that the electron should be described by a spinorial field, the photon by a vector field and the Higgs by a scalar. This is the reason that so many different fields are required in the Standard Model.

On the other hand, this picture changes completely with strings. Particles are not completely different. Rather, now they are perceived as different manifestations of the same object, the string. The fundamental object is the string and we have only two kinds of strings, open and closed. Therefore, all we need is a string field for open strings and another one for closed strings. A string field theory should contain at most just two fields. This is a major difference from point particle field theories.

2.2 Open Bosonic String Field Theory

2.2.1 The free theory

Let us start by formulating the simplest possible string field theory, that for open bosonic strings, living in a flat Minkowski spacetime, in the light cone gauge. We follow [40]. Let us denote the string field as $\Phi[X(\sigma)]$, a functional of the open string coordinates $X(\sigma)$. We take the range of σ to be in the interval $[0, \pi]$.

The reason we choose the light cone gauge is the following. Suppose that $\mathcal{L}$ is the Lagrangian of the full theory. The conjugate momentum would be

$$\Pi[X, \sigma] = \frac{\delta \mathcal{L}}{\delta \left(\frac{\delta \Phi}{\delta X^0}\right)}. \tag{2.1}$$

$X^0(\sigma)$ represents the infinity of "times" along the string and this is the reason that Π has an explicit σ dependance. However, this prevents an one to one correspondence between the string field and its momentum. By choosing the light cone gauge[1], we solve this problem.

The first step is to make a change in the coordinate system, going to light cone coordinates

$$X^{\pm} = \frac{1}{\sqrt{2}}\left(X^0 \pm X^{D-1}\right). \tag{2.2}$$

We will denote the rest of the coordinates as X^i or $\vec{X}$ and we will refer to them as the transverse coordinates. The second step is to impose the gauge condition

$$X^+ = 2\alpha' p^+ \tau. \tag{2.3}$$

This effectively fixes a common time along the string. No more infinite times to choose from in (2.1). Using the Virasoro constraints,

$$\left(\dot{X} \pm X'\right)^2 = 0, \tag{2.4}$$

we can determine X^-, up to an integration constant, in terms of p^+ and X^i. Therefore the only degrees of freedom left are X^i, x_0^- (or equivalently p^+) and $X^+ \equiv x^+$. Furthermore, these are the only physical degrees of freedom.

The physical meaning of the field is that it is the wavefunction of the object under consideration. So, the string field is the wavefunction of a string and in the light cone gauge, it is required to obey a Schrödinger type equation,

$$H\Phi[X] = i\frac{\partial}{\partial X^+}\Phi[X]. \tag{2.5}$$

The Hamiltonian of a single string is

$$H_\tau = \pi\alpha' \int_0^\pi d\sigma \left(\vec{P}^2 + \frac{1}{(2\pi\alpha')^2}\vec{X}'^2\right) \tag{2.6}$$

and it generates translations in τ. The Hamiltonian that generates translations in x^+ is $H_\tau/(2\alpha' p^+)$ and it is this Hamiltonian that enters (2.5). We write

$$H = \frac{\pi}{2p^+}\int_0^\pi d\sigma \left(\vec{P}^2 + \frac{1}{(2\pi\alpha')^2}\vec{X}'^2.\right). \tag{2.7}$$

[1]The light cone gauge was first introduced in string theory in [63].

As is usual in quantum mechanics, in the position representation, the position $\vec{X}$ is a multiplicative operator, while the momentum $\vec{P}$ is a differential operator, in our case,

$$\vec{P} = -i\frac{\delta}{\delta \vec{X}}. \tag{2.8}$$

Equation (2.5) is a functional differential equation. We can convert this into an usual partial differential equation with infinite variables.

For an open string, the mode expansion[2] is

$$X(\sigma) = x_0 + \sqrt{2}\sum_{n=1}^{\infty} x_n \cos n\sigma. \tag{2.9}$$

Utilizing this mode expansion (2.9), we can write for H,

$$H = \frac{1}{2p^+}\left\{H_0 + \sum_{n=1}^{\infty} H_n\right\}, \tag{2.10}$$

where

$$H_0 = -\frac{\partial^2}{\partial {x_0^i}^2} \tag{2.11}$$

and

$$H_n = -\frac{\partial^2}{\partial {x_n^i}^2} + \frac{n^2}{(2\alpha')^2}(x_n^i)^2. \tag{2.12}$$

Notice that each term H_n of the Hamiltonian is the Hamiltonian of a simple harmonic oscillator (except for the zero mode, H_0, which is of course the Hamiltonian of a freely propagating particle).

The equation of motion (2.5) reads in terms of the modes

$$\frac{1}{2p^+}\left\{-\frac{\partial^2}{\partial {x_0^i}^2} + \sum_{n=1}^{\infty}\left[-\frac{\partial^2}{\partial {x_n^i}^2} + \frac{n^2}{(2\alpha')^2}(x_n^i)^2\right]\right\}\Phi = i\frac{\partial \Phi}{\partial x^+} \tag{2.13}$$

and can be solved by separating variables. The solution is

$$\begin{aligned} \Phi[x^+, x_0^-, \vec{X}(\sigma)] &= \int \frac{d\vec{p}}{(2\pi)^{D-2}} \int_0^{\infty} \frac{dp^+}{2\pi} e^{i\left(\vec{p}_0\cdot\vec{x}_0 - x^+p^- - x_0^-p^+\right)} \\ & \sum_{\{n_l^i\}} A\left(p^+, \vec{p}_0, \{n_l^i\}\right) f_{\{n_l^i\}}(x_l^i) + h.c., \end{aligned} \tag{2.14}$$

where we have defined

$$f_{\{n_l^i\}}(x_l^i) = \prod_{l=1}^{\infty} \varphi_{l,\{n_l^i\}}(x_l^i) \tag{2.15}$$

[2] See also Appendix A for more details.

and

$$\varphi_{l,\{n_l^i\}}(x_l^i) = \prod_{i=1}^{D-2} C_o\left(\{n_l^i\}\right) H_{\{n_l^i\}}\left(\sqrt{\frac{l}{2\alpha'}}x_l^i\right) e^{-l(x_l^i)^2/(4\alpha')}. \tag{2.16}$$

$H_n(x)$ are Hermite polynomials and

$$C_o\left(\{n_l^i\}\right) = \sqrt{\frac{\sqrt{l/(2\alpha')}}{2^{n_l^i}(n_l^i!)\sqrt{\pi}}} \tag{2.17}$$

is the usual normalization for the eigenfunctions of the harmonic oscillator. The fact that the string field is a superposition of simple harmonic oscillators should not be a surprise. For the energy we have

$$p^- = \frac{1}{2p^+}\left\{\vec{p}_0^{\,2} + \frac{1}{\alpha'}\sum_{i,l} l\left(n_l^i + \frac{1}{2}\right)\right\}. \tag{2.18}$$

At this point, an explanation about our notation is in order. The index i enumerates the transverse directions and its range is $i = 1, \ldots, D-2$. The index l enumerates the excitation levels and its range is $l = 1, \ldots, \infty$. The integer n_l^i is the occupation number in the i-th direction at the l-th level. The notation $\{n_l^i\}$ stands for all possible n_l^is, in other words, $\{n_l^i\}$ should be read as $n_1^1, n_2^1, \ldots, n_1^2, n_2^2, \ldots$ In that sense, there is no free i and l index from $\{n_l^i\}$. We have introduced this notation so that we can make the mode expansion for the string field more transparent and easier to read. If however, the reader is still confused, we write here the mode expansion for the string field without residing on intermediate quantities. It is

$$\begin{aligned}\Phi[x^+, x_0^-, \vec{X}(\sigma)] \;=\; & \int \frac{d\vec{p}}{(2\pi)^{D-2}} \int_0^\infty \frac{dp^+}{2\pi} e^{i\left(\vec{p}_0\cdot\vec{x}_0 - x^+p^- - x_0^-p^+\right)} \\ & \sum_{\{n_l^i\}} A\left(p^+, \vec{p}_0, \{n_l^i\}\right) \prod_{l=1}^{\infty}\prod_{i=1}^{D-2} \\ & C_o\left(\{n_l^i\}\right) H_{\{n_l^i\}}\left(\sqrt{\frac{l}{2\alpha'}}x_l^i\right) e^{-l(x_l^i)^2/(4\alpha')} + h.c.,\end{aligned} \tag{2.19}$$

where of course C_o are given by (2.17). However, we will make frequent use of this notation throughout the thesis, as we believe that it is conceptually clear and easier to handle.

From the first quantised theory of strings, we know that the string can manifest itself as a massive scalar (with negative mass, the tachyon), a massless gauge boson, a massive spin 2 particle and so on. Within the string field theory framework, the corresponding fields for these particle states of the string are obtained by multiplying the string field with suitable eigenfunctions $f_{\{n_l^i\}}(x_l^i)$ and integrating over $\{x_l^i\}$. In other words, the component fields are

$$C_{\{n_l^i\}}(x^+, x_0^-, \vec{x}_0) = \int \prod_{l=1}^{\infty} \prod_{i=1}^{D-2} \left(dx_l^i f_{n_l^i}(x_l^i) \right) \Phi[x^+, x_0^-, \vec{X}(\sigma)]. \tag{2.20}$$

We can quantise the theory canonically, by promoting the string field to an operator and imposing suitable commutation relations. The equal time commutation relations are

$$\left[\Phi[x^+, x_0^-, \vec{X}(\sigma)], \Phi[x^+, y_0^-, \vec{Y}(\sigma)]\right] = \delta(x^- - y^-) \prod_\sigma \delta\left[\vec{X}(\sigma) - \vec{Y}(\sigma)\right]. \tag{2.21}$$

Equivalently, this means that

$$\left[A(p^+, \vec{p}_0, \{n_l^i\}), A^\dagger(q^+, \vec{q}_0, \{m_k^i\})\right] = (2\pi)^{D-1} \delta(p^+ - q^+) \delta(\vec{p}_0 - \vec{q}_0) \delta_{\{n_l^i\},\{m_k^j\}}. \tag{2.22}$$

Clearly, $A, A^\dagger$ are annihilation and creation operators. Their function is to destroy (or create) an entire string in the appropriate modes. They must not be confused with the creation/annihilation operators of the first quantised theory that create (destroy) oscillating modes in one single string. In fact, we make the identification

$$A(p^+, \vec{p}_0, \{n_l^i\})|0\rangle \longleftrightarrow \prod_{l,i} \left(a_l^{i\dagger}\right)^{n_l^i} |p^+, \vec{p}_0\rangle \tag{2.23}$$

between the 1-string states of the field theory Hilbert space and the states of the first quantised string Hilbert space.

The calculation of the free propagator is straight forward to perform. Using the identity

$$\sum_n H_n(x) H_n(y) \frac{T^n}{2^n n!} = \frac{1}{\sqrt{1 - T^2}} \exp\left[\frac{2xyT - T^2(x^2 + y^2)}{1 - T^2}\right] \tag{2.24}$$

for the Hermite polynomials, the result is

$$\left[\Phi(x^+, x_0^-, \vec{X}), \Phi(y^+, y_0^-, \vec{Y})\right] = G_{open}^{bosonic}(X; Y) - \{x \leftrightarrow y\}, \tag{2.25}$$

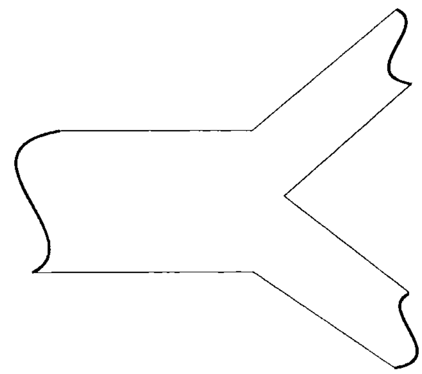

Figure 2.1: Schematic representation of the three string interaction for open strings.

where

$$
\begin{aligned}
G^{bosonic}_{open}(X;Y) &= \int_0^\infty \frac{dp^+}{2\pi}\left(-\frac{ip^+}{2\pi\Delta x^+}\right)^{(D-2)/2} \exp\left(\frac{i\Delta\vec{x}_0{}^2 p^+}{2\Delta x^+}\right) e^{-i\Delta x^- p^+} \\
&\quad \prod_{l=1}^{\infty}\prod_{i=1}^{D-2}\sqrt{\frac{l}{2\pi\alpha'}}\sqrt{\frac{1}{2i\sin\frac{l\Delta x^+}{2\alpha' p^+}}} \\
&\quad \exp\left\{\frac{l/(2\alpha')}{2i\sin\frac{l\Delta x^+}{2\alpha' p^+}}\left(2x^i_l y^i_l - \left((x^i_l)^2+(y^i_l)^2\right)\cos\frac{l\Delta x^+}{2\alpha' p^+}\right)\right\}.
\end{aligned}
\tag{2.26}
$$

In the above we have abbreviated $\Delta x^+ \equiv x^+ - y^+$, $\Delta x^- \equiv x_0^- - y_0^-$, $\Delta\vec{x}_0 \equiv \vec{x}_0 - \vec{y}_0$.

Notice that this is an infinite product of simple harmonic oscillator propagators, corresponding to the internal oscillations of the string, times the propagator of a freely moving particle, corresponding to the centre of mass of the string. This is something expected from the form of the string field, (2.14).

2.2.2 The 3-string interaction

The simplest way strings can interact is by splitting and joining at their endpoints. This process involves three strings, hence the name 3-string interaction. Figure (2.1) shows a single string propagating and then splitting into two strings, if we take the (light cone) time running from left to right. The reverse process is the merging of two strings into one (time now in figure (2.1) is running from right to left). The worldsheet for such a process looks like figure (2.2).

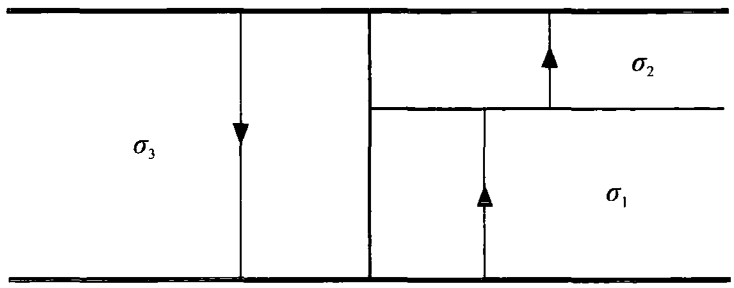

Figure 2.2: The worldsheet for the open 3-string vertex. Strings 1 and 2 merge to form string 3. The arrows indicate the way we have parameterized each string.

Let $r = 1, 2, 3$ enumerate the three strings[3]. Strings 1 and 2 are incoming and string 3 is outgoing. We have taken the parametrization of the third string to run in the opposite way than that of strings 1 and 2. This means that

$$\alpha_1 + \alpha_2 + \alpha_3 = 0 \tag{2.27}$$

with $\alpha_1, \alpha_2 > 0$ and $\alpha_3 < 0$. We have also defined $\alpha_{(r)} \equiv 2p^+_{(r)}$ and we introduce a common σ for the entire strip. It is then

$$\sigma_1 = \sigma, \text{ for } 0 \leq \sigma \leq \pi\alpha_1, \tag{2.28}$$

$$\sigma_2 = \sigma - \pi\alpha_1, \text{ for } \pi\alpha_1 \leq \sigma \leq \pi(\alpha_1 + \alpha_2), \tag{2.29}$$

$$\sigma_3 = \pi(\alpha_1 + \alpha_2) - \sigma, \text{ for } 0 \leq \sigma \leq \pi(\alpha_1 + \alpha_2). \tag{2.30}$$

It will be convenient to take the mode expansion of each string to be

$$X^i_{(r)}(\sigma) = \left(x^i_{(r),0} + 2\sum_{n=1}^{\infty} x^i_{(r),n} \cos\frac{n\sigma}{\alpha_r} \right) \Theta_{(r)}, \tag{2.31}$$

where

$$\Theta_{(1)} = \vartheta(\pi\alpha_1 - \sigma), \tag{2.32}$$

$$\Theta_{(2)} = \vartheta(\sigma - \pi\alpha_1), \tag{2.33}$$

$$\Theta_{(3)} = \Theta_{(1)} + \Theta_{(2)} = 1. \tag{2.34}$$

[3]Our presentation is based on [45] and we refer the reader there for the details. For the 3-string vertex see also [38], [39], [43], [42], [44].

ϑ is the unit step function. Then, assuming that the interaction takes place at time $x^+ = 0$, the 3-string interaction term of the string field Hamiltonian reads

$$\begin{aligned} H_3^{open} &= g \int \prod_{r=1}^{3} \left(d\alpha_r \mathcal{D}\vec{X}_{(r)} \Phi(\alpha_r, \vec{X}_{(r)}(\sigma)) \right) \delta(\alpha_1 + \alpha_2 + \alpha_3) \\ & \quad \delta\left[\vec{X}_{(1)}(\sigma) + \vec{X}_{(2)}(\sigma) - \vec{X}_{(3)}(\sigma) \right] \mu(\alpha_1, \alpha_2, \alpha_3). \end{aligned} \tag{2.35}$$

g is the string coupling constant and μ is an integration measure. This interaction is the simplest we could have and all that it demands is that the worldsheet remains continuous as the strings break and merge. Notice that it is a generalization of the ϕ^3 interaction of point-particle field theory.

Alternatively (and by a simple Fourier transformation), we can write H_3^{open} in momentum space,

$$\begin{aligned} H_3^{open} &= g \int \prod_{r=1}^{3} \left(d\alpha_r \mathcal{D}\vec{P}_{(r)} \tilde{\Phi}(\alpha_r, \vec{P}_{(r)}) \right) \delta(\alpha_1 + \alpha_2 + \alpha_3) \\ & \quad \delta\left[\sum_{r=1}^{3} \vec{P}_{(r)}(\sigma) \right] \mu(\alpha_1, \alpha_2, \alpha_3). \end{aligned} \tag{2.36}$$

We can also replace the single delta functional with the product of two,

$$\delta\left[\sum_{r=1}^{3} \vec{P}_{(r)}(\sigma) \right] = \delta\left[\vec{P}_{(1)} + \vec{P}_{(3)}\Theta_1 \right] \delta\left[\vec{P}_{(2)} + \vec{P}_{(3)}\Theta_2 \right]. \tag{2.37}$$

The two choices are equivalent.

The delta functionals are defined as infinite products over the Fourier modes of their arguments. Specifically, for the momentum delta functional we have that

$$\begin{aligned} \delta\left[\sum_{r=1}^{3} P^i_{(r)}(\sigma) \right] &= \prod_{i=1}^{D-2} \delta\left(p^i_{(1),0} + p^i_{(2),0} + p^i_{(3),0} \right) \\ & \quad \prod_{m=1}^{\infty} \delta\Big(p^i_{(3),m} + \sum_{n=1}^{\infty} \left(A^{(1)}_{mn} p^i_{(1),n} + A^{(2)}_{mn} p^i_{(2),n} \right) \\ & \quad + B^{(1)}_m p^i_{(1),0} + B^{(2)}_m p^i_{(2),0} \Big). \end{aligned} \tag{2.38}$$

The As and the Bs are determined by taking Fourier components of the identity

$$\sum_{r=1}^{3} P^i_{(r)}(\sigma) = 0 \tag{2.39}$$

in the interval $0 \leq \sigma \leq \pi(\alpha_1 + \alpha_2)$. The expansion of the momenta in terms of their modes is

$$P^i_{(r)}(\sigma) = \frac{1}{\pi|\alpha_r|}\left(p^i_{(r),0} + 2\sum_{n=1}^{\infty} p^i_{(r),n}\cos\frac{n\sigma_r}{\alpha_r}\right)\Theta_r. \tag{2.40}$$

Defining β to be

$$\beta = \frac{\alpha_1}{\alpha_3}, \tag{2.41}$$

we have that

$$A^{(1)}_{mn} = -\frac{2}{\pi}\sqrt{mn}(-1)^{m+n}\frac{\beta\sin(m\pi\beta)}{n^2 - m^2\beta^2}, \tag{2.42}$$

$$A^{(2)}_{mn} = -\frac{2}{\pi}\sqrt{mn}(-1)^{m}\frac{(\beta+1)\sin(m\pi\beta)}{n^2 - m^2(\beta+1)^2}, \tag{2.43}$$

$$B^{(1)}_m = -\alpha_2 B_m, \tag{2.44}$$

$$B^{(2)}_m = \alpha_1 B_m, \tag{2.45}$$

where we have defined

$$B_m = -\frac{2}{\pi}\frac{\alpha_3}{\alpha_1\alpha_2}m^{-3/2}(-1)^m\sin(m\pi\beta). \tag{2.46}$$

It will be very convenient to have the interaction written in the oscillator basis representation. For that purpose, we start with the string field in the momentum representation. We obtain it by Fourier transforming (2.14) and it is

$$\tilde{\Phi}[P] = \sum_{\{n^i_l\}} A(p^+, \vec{p}_0, \{n^i_l\})\prod_{l,i}\psi_{\{n^i_l\}}(p^i_l) + h.c. \tag{2.47}$$

$\psi_n(p)$ is the n-th oscillator wavefunction in the momentum representation (see also (2.16)). Inserting this expansion into (2.36) we have that, suppressing integrations,

$$H_3^{open} = \sum_{\{n^i_{(1),l}\},\{n^i_{(2),l}\},\{n^i_{(3),l}\}} A(1)A(2)A(3)C(\{n^i_{(1),l}\},\{n^i_{(2),l}\},\{n^i_{(3),l}\}) + \ldots, \tag{2.48}$$

where

$$C(\{n^i_{(1),l}\},\{n^i_{(2),l}\},\{n^i_{(3),l}\} = \mu\int\prod_{l,i} dp^i_{(1),l}dp^i_{(2),l}\psi_{\{n^i_{(1),l}\}}\psi_{\{n^i_{(2),l}\}}\psi_{\{n^i_{(3),l}\}}. \tag{2.49}$$

We have performed all the $p^i_{(3),0}$, $p^i_{(3),n}$ integrations by means of the delta functions (2.39). The dots in (2.48) stand for terms of the form $AAA^\dagger$, e.t.c.

Now recall that the momentum eigenfunction of the harmonic oscillator is

$$\psi_n(p) = \langle n|p\rangle, \tag{2.50}$$

where of course $\langle n|$ is a number basis state and $|p\rangle$ is a momentum eigenstate. $a, a^\dagger$ are of course the ladder operators of the harmonic oscillator. Given that

$$|p\rangle \propto \exp\left(-\frac{1}{4}p^2 + pa^\dagger - \frac{1}{2}a^\dagger a^\dagger\right)|0\rangle, \tag{2.51}$$

we can write for C

$$C(\{n^i_{(1),l}\},\{n^i_{(2),l}\},\{n^i_{(3),l}\}) = \langle\{n^i_{(1),l}\},\{n^i_{(2),l}\},\{n^i_{(3),l}\}|H_3\rangle. \tag{2.52}$$

It is

$$|H_3\rangle = \mu\int\prod_{l,i} dp^i_{(1),l}dp^i_{(2),l}\exp\left[\sum_{r,l,i}\left(-\frac{1}{4}\left(p^i_{(r),l}\right)^2 + p^i_{(r),l}a^{i\dagger}_{(r),l} - \frac{1}{2}a^{i\dagger}_{(r),l}a^{i\dagger}_{(r),l}\right)\right]|0\rangle, \tag{2.53}$$

where all the $p^i_{(3),n}$ have been integrated out using the delta functions.

This is just an infinite product of Gaussian integrals, that results to

$$|H_3\rangle = \mu(\det\Gamma)^{-(D-2)/2}\exp\left[\frac{1}{2}\sum_{r=1}^{3}a^{i\dagger}_{(r)}a^{i\dagger}_{(r)} - W^i\Gamma^{-1}W^i\right]|0\rangle, \tag{2.54}$$

where

$$W^i = \sum_{r=1}^{3}A_{(r)}a^{i\dagger}_{(r)} + \frac{1}{2}\left(p^i_{(1)}B^{(1)} + p^i_{(2)}B^{(2)}\right), \tag{2.55}$$

$$\Gamma = \sum_{r=1}^{3}A^{(r)}A^{(r)T}, \tag{2.56}$$

$$A^{(3)}_{mn} = \delta_{mn}. \tag{2.57}$$

The 3-string vertex can be rewritten as

$$\begin{aligned}|H_3\rangle &= \mu(\det\Gamma)^{-(D-2)/2}\exp\left[\frac{1}{2}\sum_{r,s=1}^{3}\sum_{m,n=1}^{\infty}\alpha^i_{(r),-m}\bar{N}^{rs}_{mn}\alpha^i_{(s),-n}\right.\\ &\left.+\sum_{r=1}^{3}\sum_{m=1}^{\infty}\bar{N}^r_m\alpha^i_{(r),-m}\mathbb{P} + K\mathbb{P}^2\right]|0\rangle,\end{aligned} \tag{2.58}$$

where

$$\bar{N}^{rs}_{mn} = C^{-1}_{mn}\delta_{rs} - \frac{2}{\sqrt{mn}}\left(A^{(r)T}\Gamma^{1}A^{(r)}\right)_{mn}, \tag{2.59}$$

$$\bar{N}^r_m = -\frac{1}{\sqrt{m}}(A^{(r)T}\Gamma^{-1}B)_m, \tag{2.60}$$

$$K = -\frac{1}{4}B\Gamma B \tag{2.61}$$

and

$$\mathbb{P}^i = \alpha_1 p^i_{(2),0} - \alpha_2 p^i_{(1),0}. \tag{2.62}$$

This can be further simplified. It can been shown (see [45]) that

$$\bar{N}^{rs}_{mn} = -\frac{mn\alpha_1\alpha_2\alpha_3}{n\alpha_r + m\alpha_s}\bar{N}^r_m\bar{N}^s_n. \tag{2.63}$$

Furthermore, we define

$$\bar{\phi}^{(r)}_m = \phi_m\left(-\frac{\alpha_{r+1}}{\alpha_r}\right)e^{m\tau_0/\alpha_r}, \tag{2.64}$$

$$\phi_m(x) = \frac{1}{m!}\frac{\Gamma(mx)}{\Gamma(mx+1-m)}, \tag{2.65}$$

$$\tau_0 = \sum_{r=1}^{3}\alpha_r \ln|\alpha_r|. \tag{2.66}$$

With their aid it can been shown that

$$\bar{N}^r_m = \frac{1}{\alpha_r}\bar{\phi}^{(r)}_m \tag{2.67}$$

and that

$$K = -\frac{\tau_0}{2\alpha_1\alpha_2\alpha_3}. \tag{2.68}$$

Then, the 3-string vertex can be written as

$$|H_3\rangle = \exp\left\{\frac{1}{2}\sum \alpha^i_{(r),-m}\bar{N}^{rs}_{mn}\alpha^i_{(s),-n} + \mathbb{P}^i\sum \bar{N}^r_m\alpha^i_{(r),-m} + \tau_0\sum\frac{\frac{1}{2}(p^i_{(r),0})^2-1}{\alpha_r}\right\}|0\rangle. \tag{2.69}$$

For this, it is necessary to choose the measure μ to be

$$\mu(\alpha_1,\alpha_2,\alpha_3) = (\det\Gamma)^{(D-2)/2}\exp\left(\tau_0\sum_{r=1}^{3}\frac{1}{\alpha_r}\right), \tag{2.70}$$

so that the determinants cancel. This in turn is required for the vertex to be Lorentz invariant.

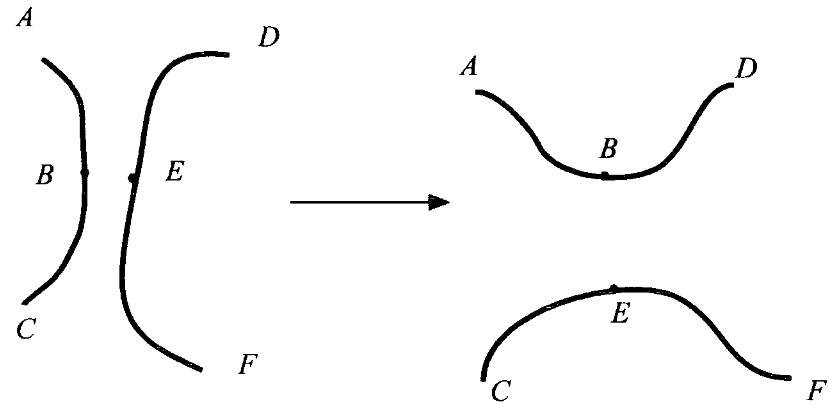

Figure 2.3: Schematic representation of four open strings interacting at an internal point (points B, E are the interaction points).

2.2.3 The 4-string interaction

Another way for open strings to interact is by joining and splitting at some interior point. Two strings propagate freely until they come in contact with each other. Then two new strings can emerge. Figure (2.3) shows the process.

We take the parametrization of each string to be $0 \leq \sigma_r \leq \pi|\alpha_r|$. Their parameters are related as follows

$$\sigma_1 = \sigma_4, \text{ for } 0 < \sigma_4 < \pi|\alpha_4|, \tag{2.71}$$

$$\sigma_1 = \sigma_2 + \pi(\alpha_1 - |\alpha_3|), \text{ for } 0 < \sigma_2 < \pi\alpha_2, \tag{2.72}$$

$$\sigma_1 = \sigma_3 + \pi(\alpha_1 - |\alpha_3|), \text{ for} 0 < \sigma_3 < \pi|\alpha_3|, \tag{2.73}$$

with $\alpha_1, \alpha_2 > 0$ and $\alpha_3, \alpha_4 < 0$. Of course,

$$\sum_{r=1}^{4} \alpha_r = 0. \tag{2.74}$$

The interaction term in the Hamiltonian of the theory is

$$\begin{aligned} H_4^{open} &= \frac{1}{2}g^2 \int \prod_{r=1}^{4} \left(d\alpha_r \mathcal{D}\vec{X}_{(r)} \Phi(\alpha_r, \vec{X}_{(r)}(\sigma)) \right) \delta(\sum_{r=1}^{4} \alpha_r) \\ & \int_{\pi(\alpha_1 - \alpha_3)}^{\pi|\alpha_4|} d\sigma_0 \prod_{\sigma_1} \delta \left[\vec{X}_{(1)}(\sigma_1) - \vec{X}_{(3)}(\sigma_3)\vartheta(\sigma_1 - \sigma_0) - \vec{X}_{(4)}(\sigma_4)\vartheta(\sigma_0 - \sigma_1) \right] \\ & \prod_{\sigma_2} \delta \left[\vec{X}_{(2)}(\sigma_2) - \vec{X}_{(3)}(\sigma_3)\vartheta(\sigma_0 - \sigma_1) - \vec{X}_{(4)}(\sigma_4)\vartheta(\sigma_1 - \sigma_0) \right]. \end{aligned} \tag{2.75}$$

We can write the vertex in the oscillator basis, in a similar way as for the 3-string vertex. We will not pursue this any further here. Instead, we refer the interested reader to the literature, [40].

2.3 Closed Bosonic String Field Theory

2.3.1 The free theory

The construction of a string field theory for closed strings[4] is similar to the one for open strings. The light cone gauge condition now is

$$X^+ = \alpha' p^+ \tau. \tag{2.76}$$

The closed string field, which we will denote by Ψ, is required to obey the same equation of motion with the open string field, (2.5), with the Hamiltonian

$$H = \frac{\pi}{p^+}\int_0^{2\pi} d\sigma \left(\vec{P}^2 + \frac{1}{(2\pi\alpha')^2}\vec{X}'^2 . \right). \tag{2.77}$$

The solution proceeds as in the open string case and the string field expansion reads

$$\begin{aligned}\Psi[x^+, x^-, \vec{X}] &= \int \frac{d\vec{p}}{2\pi}\int_0^{\infty}\frac{dp^+}{2\pi} e^{i\left(\vec{p}_0\cdot\vec{x}_0 - x^+p^- - x_0^- p^+\right)} \\ & \sum_{\{n_l^i\},\{\tilde{n}_l^i\}} A\left(p^+, \vec{p}_0, \{n_l^i\}, \{\tilde{n}_l^i\}\right) f_{\{n_l^i\}}(x_l^i) f_{\{\tilde{n}_l^i\}}(\tilde{x}_l^i) + h.c.\end{aligned} \tag{2.78}$$

where we have used the same abbreviation for $f_{\{n_l^i\}}$ as for the open string field, (2.15), and now

$$\varphi_{l,\{n_l^i\}} = \prod_{i=1}^{D-2} C_c(n_l^i) H_{\{n_l^i\}}\left(\sqrt{\frac{l}{\alpha'}}x_l^i\right)\exp\left(-\frac{l(x_l^i)^2}{2\alpha'}\right). \tag{2.79}$$

The tilded expressions are exactly the same. The normalization constant is

$$C_c(n_l^i) = \sqrt{\frac{\sqrt{l/\alpha'}}{2^{n_l^i}\left(n_l^i!\right)\sqrt{\pi}}}. \tag{2.80}$$

For the energy we have now

$$p^- = \frac{1}{2p^+}\left\{ \vec{p}_0^{\,2} + \frac{2}{\alpha'}\sum_{l,i}\left[l\left(n_l^i + \tilde{n}_l^i + 1\right)\right]\right\}. \tag{2.81}$$

[4]We follow [41] in this section.

Notice that the now we have double the oscillating modes compared with the open string case.

The theory can be quantised in a similar manner to the open string theory. The equal time commutation relations for the closed string field are the same with the open string field, (2.21). In terms of the string creation/annihhilation operators, they are

$$\begin{aligned}\left[A(p^+,\vec{p}_0,\{n_l^i\},\{\tilde{n}_l^i\}),A^\dagger(q^+,\vec{q}_0,\{m_k^i\},\{\tilde{m}_k^j\})\right] &= (2\pi)^{D-1}\delta(p^+-q^+)\delta(\vec{p}_0-\vec{q}_0)\\ &\quad \delta_{\{n_l^i\},\{m_k^j\}}\delta_{\{\tilde{n}_l^i\},\{\tilde{m}_k^j\}}. \end{aligned} \tag{2.82}$$

The calculation of the propagator proceeds in exactly the same way with the open string case and it should be no surprise that it turns out to be

$$[\Psi[X],\Psi[Y]] = G^{bosonic}_{closed}(X;Y) - \{x \leftrightarrow y\}, \tag{2.83}$$

where now

$$\begin{aligned}G^{bosonic}_{closed}(X;Y) &= \int_0^\infty \frac{dp^+}{2\pi}\left(-\frac{ip^+}{2\pi\Delta x^+}\right)^{(D-2)/2}\exp\left(\frac{i\Delta\vec{x}_0{}^2p^+}{2\Delta x^+}\right)e^{-i\Delta x^- p^+}\\ &\quad \prod_{l=1}^{\infty}\prod_{i=1}^{D-2}\left(\frac{l}{\pi\alpha'}\right)\frac{1}{2i\sin\frac{l\Delta x^+}{2\alpha' p^+}}\\ &\quad \exp\left\{\frac{l/\alpha'}{2i\sin\frac{l\Delta x^+}{\alpha' p^+}}\left(2x_l^iy_l^i-\left((x_l^i)^2+(y_l^i)^2\right)\cos\frac{l\Delta x^+}{\alpha' p^+}\right)\right\}\\ &\quad \exp\left\{\frac{l/\alpha'}{2i\sin\frac{l\Delta x^+}{\alpha' p^+}}\left(2\tilde{x}_l^i\tilde{y}_l^i-\left((\tilde{x}_l^i)^2+(\tilde{y}_l^i)^2\right)\cos\frac{l\Delta x^+}{\alpha' p^+}\right)\right\}\end{aligned} \tag{2.84}$$

For the σ parametrization, we have chosen arbitrarily a point to be the origin, $\sigma = 0$. But obviously, any point along the string can serve this role. This means that the theory should be invariant under rigid σ translations $\sigma \longrightarrow \sigma + \sigma_0$, where σ_0 an arbitrary number. For the string field, that means that it should obey the constraint

$$\Psi[X(\sigma)] = \Psi[X(\sigma+\sigma_0)]. \tag{2.85}$$

These translations are generated by the operator

$$\int d\sigma\left(i\vec{X}'(\sigma)\cdot\vec{P}(\sigma)\right). \tag{2.86}$$

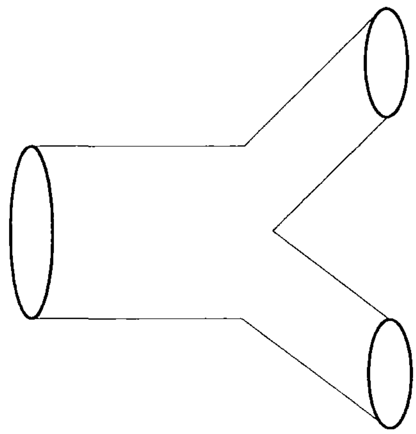

Figure 2.4: Schematic representation of a 3-string interaction for closed strings.

In terms of the modes, this is just

$$\left(\tilde{N} - N\right) \tag{2.87}$$

where

$$N = \sum_{n=1}^{\infty} \alpha^i_{-n}\alpha^i_n, \tag{2.88}$$

$$\tilde{N} = \sum_{n=1}^{\infty} \tilde{\alpha}^i_{-n}\tilde{\alpha}^i_n. \tag{2.89}$$

$N, \tilde{N}$ are the number operators of the first quantised string. In other words, the operator (2.85) imposes the usual level matching condition for the closed string.

2.3.2 Interactions

There is only one way to have strictly closed strings interaction and that is when one string breaks down to two, or two closed strings merge to form a single one. Figure (2.4) shows the 3-string interactions for the case of closed strings.

The interaction term for the 3 closed string interaction is similar to the open case. The interacting term is

$$\begin{aligned} H_3^{closed} &= \kappa \int \prod_{r=1}^{3} d\alpha_r \mathcal{D}\vec{Y}_{(r)} \Psi(\alpha_r, \vec{Y}_{(r)}) \delta(\sum_{r=1}^{3} \alpha_r) \\ &\quad \delta\left[\vec{Y}_{(3)} - \vec{Y}_{(1)} - \vec{Y}_{(2)}\right] \tilde{\mu}(\alpha_1, \alpha_2, \alpha_3), \end{aligned} \tag{2.90}$$

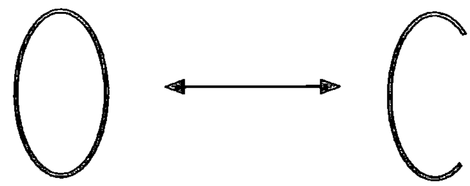

Figure 2.5: Schematic representation of a closed string breaking and forming an open string (and the reverse process).

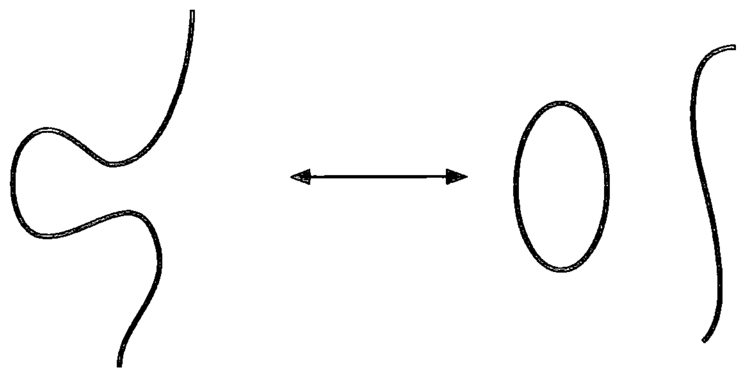

Figure 2.6: Schematic representation of an open string forming a closed string and another open string (and the reverse process).

similar to the open string case. The vertex can be analyzed in terms of the modes, the result is similar to the open string case. In fact, it is two copies of the open string vertex. For further details we refer the reader to the literature [41], [13].

In a theory combining both open and closed strings, we can have interactions involving both kinds of strings. One case is when a closed string breaks at an interior point and becomes an open string (or reversely, the end points of an open string join themselves and a closed string is formed). This is shown schematically in figure (2.5) and it is described by the interaction

$$H_2^{open-closed} = g \int \mathcal{D}\vec{X}\mathcal{D}\vec{Y} d\alpha \Phi(\alpha, \vec{X}(\sigma))\Psi(\alpha, \vec{Y}(\sigma)\delta \left[\vec{X}(\sigma) - \vec{Y}(\sigma)\right] . \tag{2.91}$$

The other case is when two interior points of an open string touch each other and a pair of a closed and an open string is formed. Figure (2.6) shows this process.

The interaction that describes it is

$$
\begin{aligned}
H_3^{open-closed} &= \frac{1}{2}g^2 \int \mathcal{D}\vec{X}_{(1)}\mathcal{D}\vec{X}_{(2)}\vec{Y} d\alpha_1 d\alpha_2 d\alpha_3 \delta(\alpha_1+\alpha_2+\alpha_3) \\
& \Phi(\alpha_1,\vec{X}_{(1)})\Phi(\alpha_2,\vec{X}_{(2)})\Psi(\alpha_3,\vec{Y}) \\
& \int_0^{\pi(\alpha_1-\alpha_3)} d\sigma_0 \delta\Big[\vec{X}_{(1)}(\sigma_1) - \vec{X}_{(2)}(\sigma_2)\vartheta(\sigma_0-\sigma_1) \\
& -\vec{X}_{(2)}(\sigma_2)\vartheta(\sigma_1-\sigma_0-\pi\alpha_3) - \vec{Y}(\sigma_3)\vartheta(-\sigma_0+\sigma_1)\theta(-\sigma_1+\sigma_0+\pi\alpha_3)\Big].
\end{aligned}
\tag{2.92}
$$

In all cases, we can write the interaction in the oscillator basis with a suitable interaction vertex. The result is similar to the open 3-string vertex, (2.58) or (2.69) and we will not repeat it here. We refer the interested reader to the literature.

One question that we have not answered is whether these are the only interactions that a full (open + closed) string theory can admit. The answer to this question is yes, but for the details we refer the read to the literature, see [18], [40].

2.4 Superstring Field Theory

Although bosonic string theory has many nice features that make it worthy studying, it has a great disadvantage. There are no fermions, which means that bosonic strings can not describe our world. We need to extend the theory by including fermions, if we are to have a more realistic theory.

There are two ways to introduce fermions in string theory. One is by introducing worldsheet spinors and imposing worldsheet supersymmetry. Then, it turns out that some of the excitations of the string behave as bosons and some others as fermions. From the spacetime point of view there is no supersymmetry, but with a suitable projection one can get it. This is the Ramond-Neuveu-Schwarz model, sometimes referred as the spinning string. As a first quantised theory it can be worked out either in the light cone gauge, or one can proceed with covariant methods of quantisation.

A different approach is to replace the Minkowski space with a flat superspace. That means that in addition to the usual (commuting) coordinates x^μ, one has anticommuting coordinates θ^A. A string living in such a space has both bosonic and fermionic excitations naturally and further more, there is spacetime supersymmetry from the beginning (no projection required). This is the Green-Schwarz model. Its disadvantage is that quantisation in a covariant way is prohibitively difficult and has not been fully achieved yet[5]. However, the theory simplifies remarkably in the light cone gauge and it can be shown that it is equivalent to the RNS model.

For our purpose, we will base our analysis on the GS superstring model. In this model, the position of the string is given by the usual coordinates $X^\mu(\sigma)$ and two anticommuting coordinates $\theta(\sigma), \tilde{\theta}(\sigma)$. It turns out that there are only five consistent superstring theories. These are:

1. Type I superstrings. They involve both open and closed strings that are non-orientable. The theory must have an $SO(32)$ gauge group symmetry, which is implemented by adding Chan-Paton factors at the end points of the open strings. There is only one spacetime supersymmetry.

2. Type II superstrings. They involve only orientable closed strings (no gauge group). They have two spacetime supersymmetries. If $\tilde{\theta}$ has the same chirality with θ, the theory is called IIB, if $\theta, \tilde{\theta}$ are of opposite chirality, the theory is called IIA.

3. Heterotic strings. They involve only closed strings. The left moving sector is the only supersymmetric one, the right moving is purely bosonic. The 16 extra dimensions of the bosonic sector (compared to the supersymmetric) are compactified and give a gauge group to the theory. The only consistent choices turn out to be $SO(32)$ and $E_8 \times E_8$.

Rather than exploring all five possibilities separately, we will discuss the open superstring in detail. The closed string case will be similar (just as the bosonic closed string was similar to the bosonic open string, remember the doubling of the modes) and will not be discussed in any detail. Heterotic strings are a bit more complicated but not significantly different.

[5] Although significant progress has been made recently with the work of Berkovits, see [52], [53], [54].

2.4.1 The free theory

The light cone gauge in the superstrings is imposed by setting

$$X^+ = 2\alpha' p^+ \tau, \tag{2.93}$$

$$\Gamma^+ \theta = 0 = \Gamma^+ \tilde{\theta}. \tag{2.94}$$

Here Γ^μ are the Dirac matrices for a ten dimensional spacetime. The choice of light cone gauge leaves a theory with a manifest $SO(8)$ spacetime symmetry, the rotations in the transverse directions. Although one can formulate the theory as such, it turns out to be more convenient to break this symmetry further, by treating two of the transverse directions separately from the other six. The reason is that the spinors θ, $\tilde{\theta}$ are Majorana-Weyl and as such they are both coordinates and conjugate momenta. We need to have distinct coordinates and momenta and this is accomplished by $SO(8) \to SO(6) \times SO(2) \sim SU(4) \times U(1)$. In return, this implies that for the transverse vectors we have the decomposition

$$A^i = \{A^I, A^L, A^R\} \tag{2.95}$$

for a transverse vector, where $I = 1, 2, \ldots, 6$ and

$$A^{R,L} = \frac{1}{\sqrt{2}} \left(A^7 \pm i A^8 \right). \tag{2.96}$$

The spinors $\theta, \tilde{\theta}$ are decomposed as

$$\theta^{\bar{A}}, \lambda^A, \tilde{\theta}^{\bar{A}}, \tilde{\lambda}^A, \tag{2.97}$$

with $A = 1, 2, 3, 4$ being a spinorial index for the $\mathbf{4}$ representation of $SU(4)$ and similarly for $\bar{A}$ for the $\bar{\mathbf{4}}$ representation. $\lambda, \tilde{\lambda}$ are conjugate momenta of $\theta, \tilde{\theta}$ respectively. From the physical point of view this treatment is not as absurd as it seems, since eventually, for a physical string theory we will require six dimensions to compactify.

The string field now is a functional, $\Phi[X(\sigma), \theta(\sigma), \tilde{\theta}(\sigma)]$ for the open string case, with a similar expression for the closed string Ψ. Often, we will use $Z(\sigma)$ to stand for all the coordinates, collectively, $Z(\sigma) = \{X(\sigma), \theta(\sigma), \tilde{\theta}(\sigma)\}$. In addition the open string field will carry group theory indices, Φ^{ab}, appropriate for the $SO(32)$ group. In practice we will often suppress them. Products then of open string fields are understood to involve a trace over the gauge group indices. The non-orientability condition is expressed as

$$\Phi^{ab}\left[X(\sigma),\theta(\sigma),\tilde{\theta}(\sigma)\right] = -\Phi^{ba}\left[X(\pi-\sigma),\theta(\pi-\sigma),\tilde{\theta}(\pi-\sigma)\right]. \tag{2.98}$$

Finally, Φ is required to be TCP-self conjugate, which means that

$$\hat{\Phi}^{ab}[X,\theta,\tilde{\theta}] = \Phi^{ba\star}[X,\theta/2,\tilde{\theta}/2]. \tag{2.99}$$

The hat means the Fourier transform with respect to the Grassmann variables only,

$$\hat{\Phi}[X,\lambda,\tilde{\lambda}] = \int \mathcal{D}\theta\mathcal{D}\tilde{\theta}\exp\left[\int_0^{\pi\alpha} d\sigma\left(\lambda^A(\sigma)\theta^{\bar{A}}(\sigma)+\tilde{\lambda}^A(\sigma)\tilde{\theta}^A(\sigma)\right)\right]\Phi[X,\theta,\tilde{\theta}]. \tag{2.100}$$

The equal time commutation relations for the open string field are

$$[\Phi[Z_1(\sigma)],\Phi[Z_2(\sigma)]] = (2\pi)^{D-1}\delta(p_1^+-p_2^+)\delta[Z_1(\sigma)-Z_2(\sigma)], \tag{2.101}$$

where

$$\delta[Z_1(\sigma)-Z_2(s)] = \delta[X_1(\sigma)-X_2(\sigma)]\delta[\theta_1(s)-\theta_2(\sigma)]\delta[\tilde{\theta}_1(\sigma)-\tilde{\theta}_2(\sigma)]. \tag{2.102}$$

The string field is again required to obey the Schrödinger type equation of motion (2.5), only that now the Hamiltonian is

$$H = \frac{\pi}{2p^+}\int_0^{\pi} d\sigma\left[-\frac{\delta^2}{\delta\vec{X}^2}+\frac{1}{(2\pi\alpha')^2}\vec{X}'^2-\frac{i}{2\pi^2\alpha'}\left(\theta'\frac{\delta}{\delta\theta}-\tilde{\theta}'\frac{\delta}{\delta\tilde{\theta}}\right)\right]. \tag{2.103}$$

With the mode decomposition

$$X^i(\sigma) = x_0^i+\sqrt{2}\sum_{n=1}^{\infty}x_n^i\cos n\sigma, \tag{2.104}$$

$$\theta^A(\sigma) = \sum_{n=-\infty}^{\infty}\theta_m^A\exp(in\sigma), \tag{2.105}$$

$$\tilde{\theta}^A(\sigma) = \sum_{n=-\infty}^{\infty}\theta_m^A\exp(-in\sigma), \tag{2.106}$$

we can convert the functional differential equation of motion into a differential equation with partial derivatives, just as we did in the bosonic case. Now, it is

$$i\frac{\partial\Phi}{\partial X^+} = \frac{1}{2p^+}\left\{-\frac{\partial^2}{\partial {x_0^i}^2}+\sum_{l=1}^{\infty}\left[-\frac{\partial^2}{\partial x_l^{i2}}+\frac{l^2}{(2\alpha')^2}{x_l^i}^2\right]+\frac{1}{2\pi\alpha'}\sum_{m=-\infty}^{\infty}m\theta_m^A\frac{\partial}{\partial\theta_m^A}\right\}\Phi. \tag{2.107}$$

Notice that the bosonic part is the same with the pure bosonic case.

From that, we can calculate the propagator for the string field. It is

$$\begin{aligned} G^{superstring}(Z_1;Z_2) &= G^{bosonic}(X_1;X_2)\cdot \prod_{m=-\infty}^{-1}\left(\theta_{(2),m}-e^{-i|m|\Delta x^+/(4\pi\alpha' p^+)}\theta_{(1),m}\right) \\ & \prod_{m=0}^{\infty}\left(\theta_{(2),m}e^{im\Delta x^+/(4\pi\alpha' p^+)}-\theta_{(1),m}\right). \end{aligned} \quad (2.108)$$

As before, we have abbreviated $\Delta x^+ = x^+ - y^+$. For the commutator of two string fields at different times, it is of course

$$[\Phi[Z_1],\Phi[Z_2]] = G^{superstring}(Z_1;Z_2) - \{x \leftrightarrow y\}. \quad (2.109)$$

2.4.2 The 3-string interaction

Interactions can be incorporated in a similar fashion as in the bosonic case. For the interaction of three open superstrings, we can define the 3-string vertex in a similar way with the bosonic case. The difference now is that the vertex is required to be continuous in both the bosonic and the Grassmann coordinates (although it will turn out that this is not enough). Or, if we decide to work in the momentum representation, it is required to conserve $P(\sigma), \lambda(\sigma), \tilde{\lambda}(\sigma)$. For a process such as that in figure (2.2), this is achieved by means of the delta functionals

$$\delta[\vec{X}_{(3)}(\sigma)-\vec{X}_{(2)}(\sigma)-\vec{X}_{(1)}(\sigma)], \quad (2.110)$$

$$\delta[\theta_{(3)}(\sigma)-\theta_{(2)}(\sigma)-\theta_{(1)}(\sigma)], \quad (2.111)$$

$$\delta[\tilde{\theta}_{(3)}(\sigma)-\tilde{\theta}_{(2)}(\sigma)-\tilde{\theta}_{(1)}(\sigma)]. \quad (2.112)$$

The 3-string vertex can be written in the oscillator basis and it takes the form

$$|H_3^s\rangle = \hat{h}|V_3^s\rangle, \quad (2.113)$$

where

$$|V_3^s\rangle = \delta(\sum_{r=1}^{3}\vec{p}_{(r),0})\delta(\sum_{r=1}^{3}\alpha_r\theta^A_{(r),0})\delta(\sum_{r=1}^{3}\alpha_r)\exp\left(E_B+E_F\right)|0\rangle \quad (2.114)$$

and $\hat{h}$ is the prefactor. E_B is the exponent of the bosonic vertex, see (2.58), while E_F is the fermionic part. It is reasonable to assume that it will have a form

$$E_F = \frac{1}{2}\sum_{m,n=1}^{\infty}\sum_{r,s=1}^{3}\theta^a_{(r),-m}X^{rs}_{mn}\theta^a_{(s),-n} + \mathbb{S}^a\sum_{m=1}^{\infty}\sum_{r=1}^{3}Y^{(r)}_m\theta^a_{(r),-m}. \tag{2.115}$$

X and Y are matrices to be determined, while $\mathbb{S}^a$ is

$$\mathbb{S}^a \equiv \alpha_1\theta^a_{(2),0} - \alpha_2\theta^a_{(1),0}. \tag{2.116}$$

It will be convenient to take the mode expansion

$$\theta^a_{(r)} = \frac{1}{\pi|\alpha_r|}\sum_{n=-\infty}^{\infty}\theta^a_{(r),n}e^{in\sigma/\alpha_r}\Theta_r, \tag{2.117}$$

$$\tilde{\theta}^a_{(r)} = \frac{1}{\pi|\alpha_r|}\sum_{n=-\infty}^{\infty}\theta^a_{(r),n}e^{-in\sigma/\alpha_r}\Theta_r. \tag{2.118}$$

Notice that $\tilde{\theta}(\sigma) = \theta(-\sigma)$. Rather than using both, we can extend the range of σ to be $-\pi|\alpha_1+\alpha_2| \leq \sigma \leq \pi|\alpha_1+\alpha_2|$ and use only $\theta(\sigma)$.

The continuity condition for the vertex reads

$$\sum_{r=1}^{3}\left(\theta^a_{(r)}(\sigma)\right)|V^s_3\rangle = 0 \tag{2.119}$$

and by taking Fourier components we have that

$$\sum_{r=1}^{3}\left(\theta^a_{(r),0}\right)|V^s_3\rangle = 0 \tag{2.120}$$

for the zero modes, and

$$\left[\theta_{(3),m} + \sum_{n=-\infty}^{\infty}\left(E^{(1)}_{mn}\theta_{(1),n} + E^{(2)}_{mn}\theta_{(2),n}\right)\right]|V^s_3\rangle = 0, \quad m \neq 0 \tag{2.121}$$

for the rest (non-zero) modes. The matrices $E^{(1)}$, $E^{(2)}$ are

$$E^{(1)}_{mn} = \frac{(-1)^{m+n}\sin(m\pi\beta)}{\pi(m\beta - n)}, \tag{2.122}$$

$$E^{(2)}_{mn} = \frac{(-1)^{m}\sin(m\pi\beta)}{\pi\left(m(\beta+1)+n\right)} \tag{2.123}$$

and it is also convenient to define

$$E^{(3)}_{mn} = \delta_{mn}. \tag{2.124}$$

β is given by (2.41).

A lengthy calculation, involving identities relating the E matrices to the A and B matrices (see [45] for the details) determines X and Y to be

$$X^{rs}_{mn} = \frac{m\alpha_s - n\alpha_r}{2\alpha_r\alpha_s}\bar{N}^{rs}_{mn} \tag{2.125}$$

and

$$Y^r_m = -\frac{m}{\alpha_r}\bar{N}^r_m. \tag{2.126}$$

The vertex (2.115) is written with the $SO(8)$ symmetry. Equivalently, it can be shown (see [48], [13]) that in the $SU(4) \times U(1)$ formalism, it takes the form

$$E_F = \sum_{r,s=1}^{3}\sum_{m,n=1}^{\infty} U^{rs}_{mn} R^r_{-mA} R^{sA}_{-n} + \sum_{r=1}^{3}\sum_{m=1}^{\infty} V^r_m R_{-mA}\Theta^A, \tag{2.127}$$

where

$$U^{rs}_{mn} = \frac{m}{\alpha_r}\bar{N}^{rs}_{mn}, \tag{2.128}$$

$$V^r_m = -(\alpha_1\alpha_2\alpha_3)\sqrt{2}\frac{m}{\alpha_r}\bar{N}^r_m \tag{2.129}$$

and we have defined

$$\Theta^A = \frac{1}{\alpha_3}\left(\theta^A_{(1),0} - \theta^A_{(2),0}\right). \tag{2.130}$$

The prefactor $\hat{h}$ comes from the fact that the vertex is required to obey the supersymmetry algebra at first order in the string coupling constant and at the same time not to spoil the continuity conditions. The exponential part is not sufficient by itself. In determining the form of the prefactor, it turns out that there is a problem with divergences at the interaction point $\pi\alpha_1$ for certain operators. Careful analysis determines the prefactor to be

$$\hat{h} = \frac{1}{2}Z^L - \sqrt{\frac{1}{2}}Z^I\rho^I_{AB}Y^AY^B + \frac{1}{3}Z^R\epsilon_{ABCD}Y^AY^BY^CY^D, \tag{2.131}$$

where

$$Z^i = \frac{1}{\sqrt{2|\alpha|}}\left(\mathbb{P}^i - \alpha\sum_{r,m}\frac{m}{\alpha_r}\bar{N}^r_m\alpha^i_{(r),-m}\right), \tag{2.132}$$

$$Y^A = \sqrt{\frac{1}{2}|\alpha|}\left(\mathbb{S}^A + \sqrt{\frac{1}{2}}\sum_{r,m}\frac{m}{\alpha_r}\bar{N}^r_m R^A_{(r),-m}\right). \tag{2.133}$$

This result for the flat background is unique. The physical meaning of the prefactor is that the 3-string interaction in the full superstring field theory has a (functional) derivative coupling. This is a direct generalisation of the derivative coupling of the Yang-Mills theory.

The extension for the closed strings as well for the other string interactions is straight forward. We will not present them here, rather we will point the interested reader to the literature, [46] [48] [64], [50], see also [13].

2.5 Summary

In this chapter we have presented a brief introduction to the field theory of strings and superstrings in the light cone gauge. After establishing the necessity of a field theory for strings and arguing why the light cone gauge is the easiest way to proceed, we examined the field theory of the bosonic string. We showed that the field can be decomposed into an infinite set of harmonic oscillator, and what is more important, it can be quantised canonically. Then we examined the 3-string interaction for the open strings, establishing the 3-string vertex and writing it in the oscillator basis. The case of the closed strings turns out to be similar, practically two copies of the open string theory. Finally, we proceeded to discuss the supersymmetric extension.

This is not a full account of string field theory. What we have tried to do here is give an overview of the subject, emphasizing on the key results that we will need in subsequent chapters, leaving the detailed calculation for the literature.

Chapter 3

String Theory in the Plane Wave Background

In this chapter we will present a short introduction on string theory in the plane wave background, emphasising on the formulation of a string field theory.

The outline is to give first a motivation and sketch the origins of the plane wave background, along with its significance for string theory. Then we will present the first quantised string in the background. Based on that, we will present the field theory of strings in this background, which is the main topic of this chapter. We conclude the chapter with a summary.

Some very good review for strings in the plane wave background are [65], [66], [67].

3.1 Origin of the plane wave background

The plane wave background, or pp-wave as it is also known in the literature, was first discovered as a solution of the type IIB supergravity, [68], [69], [70] . It has the property that it is maximally supersymmetric and that superstring theory can be solved exactly in the light cone gauge. One can do a first quantisation of the string, or proceed with a string field theory formulation. Hence its major significance. Study of string theory in a different background than the flat Minkowski is a highly non trivial matter and worthy to pursue for its own merit. It comes therefore without surprise then that it has attracted so much interest over the last years.

But the plane wave background has another important property, with far reaching consequences. Starting with an $AdS_5 \times S^5$ spacetime, a certain Penrose limit gives the plane wave background. Now the $AdS_5 \times S^5$ background, whose metric can be written as

$$ds^2 = \frac{r^2}{R^2}\left(-dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2\right) + R^2\frac{dr^2}{r^2} + R^2 d\Omega_5^2, \tag{3.1}$$

together with a 5-form field strength

$$F_5 = 4R^4\left(\cosh\rho\sinh^3\rho dt\wedge d\rho\wedge d\Omega_3 + \cos\theta\sin^3\theta d\psi\wedge d\theta\wedge d\Omega_3'\right), \tag{3.2}$$

where $R^4 = 4\pi g\alpha'^2 N$, constitutes a solution of type IIB supergravity and it is maximally supersymmetric. String theory in this background is conjectured to be dual to a $\mathcal{N} = 4$ Super Yang Mills living on the boundary of the spacetime (which is 4-dimensional). This is the AdS/CFT correspondence, see [27], [28], [29], [30], [31], [32]. The duality is a strong/weak coupling duality. By that we mean that properties of one theory in the strong coupling limit can be computed by the dual theory in the weak coupling limit. Unfortunately, so far, we have been unsuccessful to prove the conjecture and even more, we have been restricted to the supergravity limit on string theory side of the correspondence[1]. This is not surprising, since the superstring action in $AdS_5 \times S^5$, constructed in [73], [74], [75], is highly complicated.

But if a certain limit of the $AdS_5 \times S^5$ string theory results in a solvable theory, then we can proceed in the correspondence beyond the supergravity limit. One open and very interesting question is what section of the Super Yang Mills is dual to the string theory in the plane wave background. This is the BMN correspondence, first proposed by [76], see also [77] (the letters BMN stand for the names of the authors). Therefore, the plane wave provides a framework where we can study a string theory/field theory duality from both sides[2]. What is even more interesting is that unlike the AdS/CFT correspondence that is strictly strong/weak, for the BMN correspondence there is a limit where both sides are at weak coupling and hence trackable simultaneously.

[1] See however [71], [72].

[2] For a review see [78], [67].

Let us start with the metric of the $AdS_5 \times S^5$, written as

$$\frac{ds^2}{R^2} = (-dt^2 \cosh^2 \rho + d\rho^2 + \sinh^2 \rho d\Omega_3^2) + (d\psi^2 \cos^2 \theta + d\theta^2 + \sin^2 \theta d\Omega_3'^2). \quad (3.3)$$

The first part corresponds to the AdS and the second part to the S^5. $\theta \in [-\pi, \pi]$ and is the latitude of the sphere, ψ is periodic (with period 2π) and runs along an equator of the sphere. This form is the same with (3.1). Now let us make the following change of coordinates,

$$x^+ = \frac{1}{2\mu}(t + \psi), \quad x^- = \mu R^2 (t - \psi), \quad (3.4)$$

$$\rho = \frac{r}{R}, \; \theta = \frac{y}{R}. \quad (3.5)$$

Taking the limit $R \to \infty$, the metric takes the form

$$ds^2 = -2dx^+dx^- - (r^2 + y^2)\mu^2(dx^+)^2 + dr^2 + r^2 d\Omega_3^2 + dy^2 + y^2 d\Omega_3'^2. \quad (3.6)$$

The last two terms are just the metric of $\mathbb{R}^4 \times \mathbb{R}^4$, or $\mathbb{R}^8$. Therefore, we can simplify the metric and write it as

$$ds^2 = -2dx^+dx^- - \mu^2(x^i)^2(dx^+)^2 + dx^i dx^i, \quad (3.7)$$

where as before $i = 1, 2, \ldots, 8$, enumerating the transverse directions. This limit means that we are focusing our attention on a particle sitting in the centre of AdS (that would be $\rho = 0$), moving very fast (close to the speed of light) along an equator of the S^5 (described by ψ). (3.7) is the geometry that this particle effectively sees. For the 5-form field strength, we have that in this limit

$$F_5 = 4\mu dx^+ \wedge \left(dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 + dx^5 \wedge dx^6 \wedge dx^7 \wedge dx^8\right). \quad (3.8)$$

The metric (3.7) along with the field (3.8) constitutes the plane wave background that we will be working on. Although it seems that this background has a $SO(8)$ transverse symmetry, in fact this is broken down to $SO(4) \times SO(4) \times Z_2$, because of the non trivial 5-form field. The first $SO(4)$ is a rotation symmetry among the first four transverse directions, $i = 1, 2, 3, 4$, the second $SO(4)$ is a rotation symmetry among the last four transverse directions, $i = 5, 6, 7, 8$ and Z_2 interchanges the two groups. This in return will have consequences later on, when we will consider the 3-string vertex.

3.2 String theory in the plane wave background

Let us start with the study of a single closed string in the plane wave background. Our presentation will be based on [79], [80]. Although the GS superstring action is very complicated, we can impose the light cone gauge condition on it and then surprisingly, we have an action that is quadratic. The light cone gauge fixing conditions are

$$X^+ = \alpha' p^+ \tau, \tag{3.9}$$

$$\bar{\gamma}^+ \theta = 0 = \bar{\gamma}^+ \bar{\theta}. \tag{3.10}$$

$\theta, \bar{\theta}$ are complex Weyl spinors. We can replace them with two real spinors, θ^1, θ^2, with the substitution

$$\theta = \frac{1}{\sqrt{2}}(\theta^1 + i\theta^2), \; \bar{\theta} = \frac{1}{\sqrt{2}}(\theta^1 - i\theta^2). \tag{3.11}$$

Our conventions for the Γ matrices are the same with [79]. For completeness we repeat them here. Γ^μ are the 32×32 Dirac gamma matrices and γ^μ are the Dirac 16×16 gamma matrices, taken to be real and symmetric.

$$\Gamma^\mu = \begin{pmatrix} 0 & \gamma^\mu \\ \bar{\gamma}^\mu & 0 \end{pmatrix}, \tag{3.12}$$

$$\gamma^\mu = (1, \gamma^i, \gamma^9), \; \bar{\gamma}^\mu = (-1, \gamma^i, \gamma^9). \tag{3.13}$$

They are required to obey the algebra

$$\gamma^\mu \bar{\gamma}^\nu + \gamma^\nu \bar{\gamma}^\mu = 2\eta^{\mu\nu}. \tag{3.14}$$

We also define

$$\Pi = \gamma^1 \bar{\gamma}^2 \gamma^3 \bar{\gamma}^4, \; \Pi' = \gamma^5 \bar{\gamma}^6 \gamma^7 \bar{\gamma}^8. \tag{3.15}$$

For a closed string the action is

$$\begin{aligned} S &= \frac{1}{4\pi\alpha'} \int d\tau \int_0^{2\pi} d\sigma \left((\partial_\tau X^i)^2 - (\partial_\sigma X^i)^2 - m^2 (X^i)^2 \right) + \\ &+ \frac{1}{8\pi} \int d\tau \int_0^{2\pi} d\sigma \left(i(\theta^1 \bar{\gamma}^- \partial_\tau \theta^1 + \theta^1 \bar{\gamma}^- \partial_\sigma \theta^1 + \theta^2 \bar{\gamma}^- \partial_\tau \theta^2 \right. \\ &\left. -\theta^2 \bar{\gamma}^- \partial_\sigma \theta^2 - 2m\theta^1 \bar{\gamma}^- \Pi \theta^2) \right),, \end{aligned} \tag{3.16}$$

where

$$m \equiv \alpha' p^+ \mu. \tag{3.17}$$

The equations of motion that follow from this action are

$$\partial_\tau^2 X^i - \partial_\sigma^2 X^i + m^2 X^i = 0, \tag{3.18}$$

$$(\partial_\tau + \partial_\sigma)\theta^1 - m\Pi\theta^2 = 0, \tag{3.19}$$

$$(\partial_\tau - \partial_\sigma)\theta^2 + m\Pi\theta^1 = 0. \tag{3.20}$$

We impose periodic boundary condition, appropriate for the closed string.

They can be solved and we have that

$$\begin{aligned} X^i(\tau,\sigma) &= x_0^i \cos(m\tau) + \frac{1}{m} p_0^i \sin(m\tau) 2\alpha' + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{\omega_n} \left\{ \alpha_n^i e^{-i(\omega_n\tau - s)} + \right. \\ & \left. \tilde{\alpha}_n^i e^{-i(\omega_n \tau + n\sigma)} \right\}, \end{aligned} \tag{3.21}$$

$$\begin{aligned} \theta^1(\tau,\sigma) &= \cos(m\tau)\theta_0^1 + \sin(m\tau)\Pi\theta_0^2 + \sum_{n\neq 0} c_n \left\{ \theta_n^1 e^{-i(\omega_n\tau - n\sigma)} + \right. \\ & \left. i\left(\frac{\omega_n - n}{m}\right) e^{-i(\omega_n\tau + n\sigma)} \Pi\theta_n^2 \right\}, \end{aligned} \tag{3.22}$$

$$\begin{aligned} \theta^2(\tau,\sigma) &= \cos(m\tau)\theta_0^2 - \sin(m\tau)\Pi\theta_0^1 + \sum_{n\neq 0} c_n \left\{ \theta_n^2 e^{-i(\omega_n\tau + n\sigma)} - \right. \\ & \left. i\left(\frac{\omega_n - n}{m}\right) e^{-i(\omega_n\tau - n\sigma)} \Pi\theta_n^1 \right\}, \end{aligned} \tag{3.23}$$

In the above, we have defined

$$\omega_n = \begin{cases} \sqrt{n^2+m^2}, & n>0, \\ -\sqrt{n^2+m^2}, & n<0 \end{cases} \tag{3.24}$$

and

$$c_n = \frac{1}{\sqrt{1+\left(\frac{\omega_n - n}{m}\right)^2}}. \tag{3.25}$$

It should be obvious now, that we can quantise the theory without any difficulty, simply by promoting coordinates and momenta to operators and imposing the usual commutation relations. In terms of the modes, these are

$$[x_0^i, p_0^j] = i\delta^{ij}, \; [\alpha_n^i, \alpha_m^j] = \frac{1}{2}\omega_n \delta_{n+m,0}\delta^{ij} = [\tilde{\alpha}_n^i, \tilde{\alpha}_m^j], \tag{3.26}$$

$$\{\theta_n^{1a}, \theta_m^{1b}\} = \frac{1}{4}(\gamma^+)^{ab}\delta_{n+m,0} = \{\theta_n^{2a}, \theta_m^{2b}\}, \tag{3.27}$$

the rest being identically zero.

Notice that all the above formulas have the important property that in the limit $\mu \to 0$ they fall back to the flat case. This is something expected, since the plane wave metric (3.7) in that limit reduces to the Minkowski metric.

The open string case is similar, practically just (say) the left moving sector of the closed string. For the details as well as for extensions towards D-brane physics, see [81], [82], [83].

3.3 String Field Theory in the plane wave background

Since string theory in the plane wave background can be solved in the light cone gauge, we should be able to formulate a string field theory. The procedure is the same as in the flat background. The string field is a functional of the coordinates (both bosonic and fermionic), $\Psi[X, \theta^1, \theta^2]$. Following the same procedure with the flat background case, we can use the mode expansion for X, θ^1, θ^2 to write the string field as a function of the modes.

3.3.1 The free theory

For simplicity, let us analyze the free string field for open strings, suppressing for the moment the fermionic part. The string field is again required to obey the same Schrödinger type equation of motion as in the flat background case, equation (2.5), with H now being of course

$$H = \frac{\pi}{2p^+}\int_0^\pi d\sigma \left[\vec{P} + \frac{1}{(2\pi\alpha')^2}\vec{X}'^2 + \frac{m^2}{(2\pi\alpha')^2}\vec{X}^2\right]. \tag{3.28}$$

The equation of motion is

$$\frac{\pi}{2p^+}\int_0^\pi d\sigma \left[-\frac{\delta^2}{\delta\vec{X}(\sigma)^2} + \frac{1}{(2\pi\alpha')^2}\vec{X}'^2 + \frac{m^2}{(2\pi\alpha')^2}\vec{X}^2\right] = i\frac{\partial\Phi}{\partial x^+}, \tag{3.29}$$

taking

$$\vec{P}(\sigma) = -i\frac{\delta}{\delta \vec{X}(\sigma)}. \tag{3.30}$$

It is a functional differential equation, but nevertheless, we can solve it, as we did for the flat background theory.

The mode expansion for an open string (at $\tau = 0$) is

$$X^i(\sigma) = x_0^i + \sqrt{2}\sum_{l=1}^{\infty} x_l^i \cos l\sigma. \tag{3.31}$$

Then, as we did for the string field in the flat background, we can write for H,

$$H = \sum_{l=0}^{\infty} H_l, \tag{3.32}$$

where now

$$H_l = \frac{1}{2p^+}\left(-\frac{\partial^2}{\partial x_l^{i2}} + \frac{1}{(2\alpha')^2}\omega_l^2 x_l^{i2}\right). \tag{3.33}$$

For the open string it is $\omega_l = \sqrt{l^2+m^2}$, with $m \equiv 2\alpha'\mu p^+$. Notice that now the zero mode of the string does not correspond to a freely propagating particle. It is also an oscillating mode.

The equation of motion for the string field takes the form

$$\sum_{l=0}^{\infty} H_l \Phi = i\frac{\partial \Phi}{\partial x^+} \tag{3.34}$$

and can be solved by separating variables. The solution is, as for the flat case, a superposition of an infinite tower of harmonic oscillators,

$$\begin{aligned}\Phi[x^+, x_0^-, \vec{X}] &= \int_0^\infty \frac{dp^+}{2\pi} e^{-i\left(x^+p^- + x_0^- p^+\right)} \\ &\quad \sum_{\{n_l^i\}} A\left(p^+, \{n_l^i\}\right) f_{\{n_l^i\}}(x_l^i) + h.c.,\end{aligned} \tag{3.35}$$

where we have defined

$$f_{\{n_l^i\}}(x_l^i) = \prod_{l=0}^{\infty} \varphi_{l,\{n_l^i\}}(x_l^i) \tag{3.36}$$

and

$$\varphi_{l,\{n_l^i\}}(x_l^i) = \prod_{i=1}^{8} C_o\left(\{n_l^i\}\right) H_{\{n_l^i\}}\left(\sqrt{\frac{\omega_l}{2\alpha'}}x_l^i\right) e^{-\omega_l (x_l^i)^2/(4\alpha')}. \tag{3.37}$$

$H_n(x)$ are of course Hermite polynomials and

$$C_o\left(\{n_l^i\}\right) = \sqrt{\frac{\sqrt{\omega_l/(2\alpha')}}{2^{n_l^i}(n_l^i!)\sqrt{\pi}}} \tag{3.38}$$

is the usual normalization for the eigenfunctions of the harmonic oscillator. For the energy, we have that

$$p^- = \frac{1}{2\alpha' p^+} \sum_{l=0}^{\infty} \sum_{i=1}^{8} \omega_l \left(n_l^i + \frac{1}{2} \right) . \tag{3.39}$$

Notice that the zero mode in no longer freely propagating, but is included in the oscillators. This is the reason that we do not have an integration over $\vec{p}_0$ now. This fact stems from the form of the Hamiltonian in the plane wave, which in turn is due to the extra μ-term in the plane wave metric.

The theory can be canonically quantised by imposing the equal time commutator relations

$$\left[\Phi[x^+, x_0^-, \vec{X}], \Phi[x^+, y_0^-, \vec{Y}] \right] = \delta(x_0^- - y_0^-)\delta[\vec{X} - \vec{Y}] \tag{3.40}$$

for the string field. This in turn implies that

$$\left[A(p^+, \{n_l^i\}), A^\dagger(q^+, \{m_k^j\}) \right] = 2\pi\delta(p^+ - q^+)\delta_{\{n_l^i\},\{m_k^j\}} . \tag{3.41}$$

As in the flat background case, the role of the $A, A^\dagger$ is to destroy/create entire strings with the appropriate modes.

The calculation of the propagator is not hard to perform and we get

$$\left[\Phi(x^+, x_0^-, \vec{X}), \Phi(y^+, y_0^-, \vec{Y}) \right] = \tilde{G}_{open}^{bosonic}(X;Y) - \{x \leftrightarrow y\}, \tag{3.42}$$

where

$$\begin{aligned} \tilde{G}_{open}^{bosonic} &= \int_0^\infty \frac{dp^+}{2\pi} e^{-i\Delta x_0^- p^+} \prod_{l=0}^{\infty} \prod_{i=1}^{8} \sqrt{\frac{\omega_l}{2\pi\alpha'}} \sqrt{\frac{1}{2i \sin \frac{\omega_l \Delta x^+}{2\alpha' p^+}}} \\ & \exp\left\{ \frac{\omega_l/(2\alpha')}{2i \sin\left(\frac{\Delta x^+ \omega_l}{2\alpha' p^+}\right)} \left[2x_l^i y_l^i - \left((x_l^i)^2 + (y_l^i)^2\right) \cos\left(\frac{\Delta x^+ \omega_l}{2\alpha' p^+}\right) \right] \right\} \end{aligned} \tag{3.43}$$

As before, we have abbreviated $\Delta x^- = x_0^- - y_0^-$, $\Delta x^+ = x^+ - y^+$.

The extension to the case of closed strings and the inclusion of fermions is straight forward. Notice that the construction resembles remarkably the construction of the string field theory in the flat background. The important difference is that now the zero mode is no longer freely propagating but oscillating. This is due to the extra term in the metric (3.7).

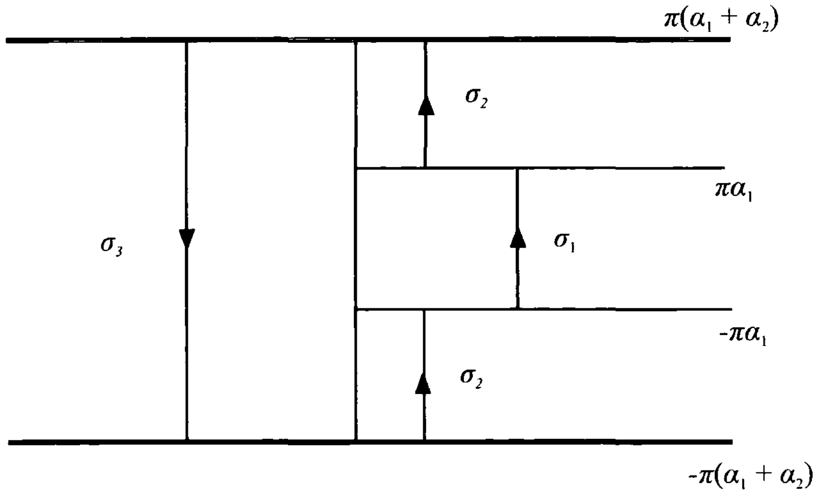

Figure 3.1: The worldsheet for the closed 3-string vertex. Strings 1 and 2 merge to form string 3. The arrows indicate the way we have parameterized each string. Identifications are implied, appropriate for closed strings.

3.3.2 Interactions

The next step, is to include interactions in the theory. As was the case in the flat background, the simplest interaction is the 3-string interaction. We will analyse the case for the closed strings and obtain the vertex in the oscillator basis. The interaction schematically is as in figure (2.4), a closed string splitting to two other closed strings, or the reverse process where two closed strings merge to form one. Figure (3.1) shows the worldsheet for the interaction.

The procedure is the same with the flat background case. The interaction is required to be continuous in the coordinates, which is secured by means of the functionals

$$\delta\left[\vec{X}_{(3)}(\sigma) - \vec{X}_{(1)}(\sigma) - \vec{X}_{(2)}(\sigma)\right], \tag{3.44}$$

$$\delta\left[\theta^1_{(3)}(\sigma) - \theta^1_{(1)}(\sigma) - \theta^1_{(2)}(\sigma)\right], \tag{3.45}$$

$$\delta\left[\theta^2_{(3)}(\sigma) - \theta^2_{(1)}(\sigma) - \theta^2_{(2)}(\sigma)\right]. \tag{3.46}$$

Alternatively, the interaction is required to conserve the momentum. We write the interacting part of the field Hamiltonian as

$$H_3 = \int d\mu_3 h_3(\alpha_r, P_{(r)}(\sigma), \lambda_{(r)}(\sigma))\Psi[1]\Psi[2]\Psi[3], \tag{3.47}$$

where

$$d\mu_3 = \prod_{r=1}^{3}\left(d\alpha_r \mathcal{D}\vec{P}_{(r)}\mathcal{D}\lambda_{(r)}\right)\delta(\sum_{r=1}^{3} a_r)\delta\left[\sum_{r=1}^{3}\lambda_{(r)}(\sigma)\right]\delta\left[\sum_{r=1}^{3}\vec{P}_{(r)}(\sigma)\right] \tag{3.48}$$

and h_3 is the prefactor.

Passing to the oscillator basis follows the same procedure as in the flat case. One then can define the 3-string vertex operator,

$$|H_3\rangle = h_3|V\rangle. \tag{3.49}$$

$|V\rangle$ takes the form

$$|V\rangle \propto |E_b\rangle|E_f\rangle|0\rangle\delta\left(\sum_{r=1}^{3}\alpha_r\right). \tag{3.50}$$

$|E_b\rangle$ is the bosonic part of the vertex and $|E_f\rangle$ is the fermionic part. h_3 is the prefactor. The meaning of each factor and the methodology of obtaining them is essentially the same as in the flat background case, although these are not enough to determine the vertex uniquely.

The bosonic part

Let us first discuss the bosonic part of the vertex. To that end, it will be convenient to take the range of parametrization for each string to be $-\pi|\alpha_r| \leq \sigma_r \leq \pi|\alpha_r|$, where $r = 1, 2, 3$ enumerates the strings and $\alpha_r \equiv \alpha' p_r^+$. We will take strings 1 and 2 to be the incoming (hence $\alpha_1, \alpha_2 > 0$) and string 3 to be the outgoing (hence $\alpha_3 < 0$). Introducing a common σ for the entire worldsheet in figure (3.1), we have

$$\begin{aligned}
\sigma_1 &= \sigma, && \text{for } -\pi\alpha_1 \leq \sigma \leq \pi\alpha_1\,, \\
\sigma_2 &= \begin{cases} \sigma - \pi\alpha_1, & \text{for } \pi\alpha_1 \leq \sigma \leq \pi(\alpha_1+\alpha_2)\,, \\ \sigma + \pi\alpha_1, & \text{for } -\pi(\alpha_1+\alpha_2) \leq \sigma \leq -\pi\alpha_1\,, \end{cases} \\
\sigma_3 &= -\sigma, && \text{for } -\pi(\alpha_1+\alpha_2) \leq \sigma \leq \pi(\alpha_1+\alpha_2)
\end{aligned} \tag{3.51}$$

It will also be convenient to take the mode expansion for the coordinates and the momenta to be

$$X^i_{(r)}(\sigma) = \left[x^i_{(r),0} + \sqrt{2}\sum_{n=1}^{\infty}\left(x^i_{(r),n}\cos\frac{n\sigma}{|\alpha_r|} + x^i_{(r),-n}\sin\frac{n\sigma}{|\alpha_r|}\right)\right]\Theta_r \tag{3.52}$$

and

$$P^i_{(r)}(\sigma) = \frac{1}{2\pi|\alpha_r|}\left[p^i_{(r),0} + \sqrt{2}\sum_{n=1}\left(p^i_{(r),n}\cos\frac{n\sigma}{|\alpha_r|} + p^i_{(r),-n}\sin\frac{n\sigma}{|\alpha_r|}\right)\right]\Theta_r, \tag{3.53}$$

respectively. Θ_r are

$$\Theta_1 = \vartheta(\pi\alpha_1 - |\sigma|), \tag{3.54}$$

$$\Theta_2 = \vartheta(|\sigma| - \pi\alpha_1), \tag{3.55}$$

$$\Theta_3 = \Theta_1 + \Theta_2 = 1, \tag{3.56}$$

with ϑ the unit step function.

Also, the momentum delta functional in (3.48) can be reexpressed as an infinite product of delta functions of the Fourier modes, by taking Fourier components of the identity $P_1 + P_2 + P_3 = 0$. To that end, we need the integrals (for $n, m > 0$ and with $\beta = \alpha_1/\alpha_3$)

$$\frac{1}{\pi\alpha_1}\int_{-\pi\alpha_1}^{\pi\alpha_1} d\sigma\cos\frac{m\sigma}{\alpha_3}\cos\frac{n\sigma}{\alpha_1} = (-1)^n\frac{2m\beta}{\pi}\frac{\sin(m\pi\beta)}{m^2\beta^2 - n^2} \equiv \tilde{X}^{(1)}_{mn}, \tag{3.57}$$

$$\frac{1}{\pi\alpha_1}\int_{-\pi\alpha_1}^{\pi\alpha_1} d\sigma\sin\frac{m\sigma}{\alpha_3}\sin\frac{n\sigma}{\alpha_1} = \frac{n}{m\beta}\tilde{X}^{(1)}_{mn}, \tag{3.58}$$

$$\frac{2}{\pi\alpha_2}\int_{\pi\alpha_1}^{\pi(\alpha_1+\alpha_2)} d\sigma\cos\frac{m\sigma}{\alpha_3}\cos\frac{n(\sigma-\pi\alpha_1)}{\alpha_2} = \frac{2m(\beta+1)}{\pi}\frac{\sin(m\pi\beta)}{m^2(\beta+1)^2 - n^2} \equiv \tilde{X}^{(2)}_{mn}, \tag{3.59}$$

$$\frac{2}{\pi\alpha_2}\int_{\pi\alpha_1}^{\pi(\alpha_1+\alpha_2)} d\sigma\sin\frac{m\sigma}{\alpha_3}\sin\frac{n(\sigma-\pi\alpha_1)}{\alpha_2} = -\frac{n}{m(\beta+1)}\tilde{X}^{(2)}_{mn}. \tag{3.60}$$

Then, we can write the delta functional as

$$\delta\left[\sum_{r=1}^{3} P_{(r)}(\sigma)\right] \propto \prod_{m\in\mathbb{Z}}\delta\left(\sum_{r=1}^{3}\sum_{n\in\mathbb{Z}} X^{(r)}_{mn}p_{(r),n}\right), \tag{3.61}$$

where we have defined

$$\tilde{X}^{(3)}_{mn} = \delta_{mn}. \tag{3.62}$$

The matrices $X^{(r)}_{mn}$ in (3.61) for all m, n are

$$X^{(r)}_{mn} \equiv \begin{cases} \tilde{X}^{(r)}_{mn}\,, & m\,,n > 0 \\ \frac{\alpha_3}{\alpha_r}\frac{n}{m}\tilde{X}^{(r)}_{-m,-n}\,, & m\,,n < 0 \\ \frac{1}{\sqrt{2}}\tilde{X}^{(r)}_{m0}\,, & m > 0\,, r \in \{1,2\} \\ 1\,, & m = 0 = n \\ 0\,, & \text{otherwise} \end{cases} \tag{3.63}$$

These matrices are related to the As and BS, (2.42), (2.43), (2.46), as follows

$$A^{(1,2)}_{mn} = \left(C^{-1/2}X^{(1,2)}C^{1/2}\right)_{mn} \tag{3.64}$$

and

$$X^{(r)}_{m0} = -\epsilon^{rs}\alpha_s\left(C^{1/2}B\right)_m . \tag{3.65}$$

The string field in the momentum representation is

$$\Psi = \sum_{\{n^i_l\}} A(p^+,\{n^i_l\}) \prod_{l\in\mathbb{Z}}\prod_{i=1}^{8}\varphi(p^i_{n^i_l}) + h.c. \tag{3.66}$$

Inserting this expansion into the interacting Hamiltonian (3.48), using the relation between momentum eigenstates $|p_l\rangle$ and the oscillator basis,

$$|p_l\rangle = \left(\frac{\omega_l\pi}{\alpha'}\right)^{-1/4}\exp\left[-\frac{\alpha'}{2\omega_l}p_l^2 + \sqrt{\frac{2\alpha'}{\omega_l}}a^\dagger_l p_l - \frac{1}{2}a^\dagger_l a^\dagger_l\right]|0\rangle, \tag{3.67}$$

the decomposition of the delta functional (3.61) and performing the resulting Gaussian integrals, we have at the end

$$|E_b\rangle = \exp(E_B^{plane\ wave})|0\rangle, \tag{3.68}$$

where

$$E_B^{plane\ wave} = \frac{1}{2}\sum_{r,s=1}^{3}\sum_{m,n} a^{i\dagger}_{(r),m}\bar{N}^{rs}_{mn}a^{i\dagger}_{(s),n}, \tag{3.69}$$

where we have defined the Neumann matrices (for $n, m > 0$) to be

$$\bar{N}^{rs}_{mn} = \delta^{rs}\delta_{mn} - 2\sqrt{\frac{\omega_{(r),m}\omega_{(s),n}}{mn}}\left(A^{(r)T}\Gamma_{-1}A^{(s)}\right)_{mn}, \tag{3.70}$$

$$\bar{N}^{rs}_{m0} = -\sqrt{2\mu\alpha_s\omega_{(r)m}}\varepsilon^{st}\alpha_t\bar{N}^r_m,\ s\in\{1,2\}, \tag{3.71}$$

$$\bar{N}^{rs}_{00} = (1-4\mu\alpha K)\left(\delta^{rs} + \frac{\sqrt{\alpha_r\alpha_s}}{\alpha_3}\right),\ r,s\in\{1,2\}, \tag{3.72}$$

$$\bar{N}^{r3}_{00} = -\sqrt{-\frac{\alpha_r}{\alpha_3}},\ r,s\in\{1,2\}. \tag{3.73}$$

It is $\alpha = \alpha_1\alpha_2\alpha_3$ and

$$\Gamma \equiv \sum_{r=1}^{3} A^{(r)}U_{(r)}A^{(r)T} \tag{3.74}$$

and

$$U_{(r)} \equiv C^{-1}(C_{(r)} - \mu a_r), \tag{3.75}$$

with $C_{mn} = m\delta_{mn}$, $\left(C_{(r)}\right)_{mn} = \omega_m \delta_{mn}$. Furthermore,

$$\bar{N}^r \equiv -C^{-1/2} A^{(r)T} \Gamma^{-1} B, \;\; K \equiv -\frac{1}{4} B^T \Gamma^{-1} B. \tag{3.76}$$

The Neumann matrices with negative indices are

$$\bar{N}^{rs}_{-m,-n} = -\left(U_{(r)} \bar{N}^{rs} U_{(s)}\right)_{mn}, \;\; m, n > 0, \tag{3.77}$$

in terms of the Neumann matrices with positive indices. The identity (2.63), combined with the explicit results (2.67), (2.68) solved the vertex completely in the flat background. Here the determination of the Neumann matrices is much more difficult, however is has been achieved, see [84], [85].

The fermionic part

Similarly, for the fermionic part of the vertex[3] we use the mode decomposition

$$\theta^a_{(r)}(\sigma) = \left[\theta^a_{(r),0} + \sqrt{2} \sum_{n=1}^{\infty} \left(\theta^a_{(r),n} \cos \frac{n\sigma}{|\alpha_r|} + \theta^a_{(r),-n} \sin \frac{n\sigma}{|\alpha_r|}\right)\right] \Theta_r \tag{3.78}$$

and

$$\lambda^a_{(r)}(\sigma) = \frac{1}{2\pi |\alpha_r|} \left[\lambda^a_{(r),0} + \sqrt{2} \sum_{n=1}^{\infty} \left(\lambda^a_{(r),n} \cos \frac{n\sigma}{|\alpha_r|} + \lambda^a_{(r),-n} \sin \frac{n\sigma}{|\alpha_r|}\right)\right] \Theta_r \tag{3.79}$$

for θ and its conjugate momentum λ respectively.

The matrix Π in the action (3.16) breaks the transverse $SO(8)$ symmetry down to $SO(4) \times SO(4)$. For that purpose it is convenient to define a new set of fermionic operators

$$\theta_{(r),n} = \frac{c_{(r),n}}{\sqrt{|\alpha_r|}} \left[\left(1 + \rho_{(r),n} \Pi\right) b_{(r),n} + \mathrm{sgn}(\alpha_r) \mathrm{sgn}(n) \left(1 - \rho_{(r),n} \Pi\right) b^\dagger_{(r),n}\right] \tag{3.80}$$

that explicitly break the $SO(8)$ symmetry. In the above, we have defined

$$\rho_{(r),n} = \rho_{(r),-n} = \frac{\omega_{(r),n} - |n|}{\mu \alpha_r}, \tag{3.81}$$

$$c_{(r),n} = c_{(r),-n} = \frac{1}{\sqrt{1 + \rho^2_{(r),n}}}. \tag{3.82}$$

[3] This section and the next are based on [86], [87], [88].

The vacuum is defined to be annihilated by all a's and b's, that is

$$a_{(r),n}|v\rangle_{(r)} = 0, \quad b_{(r),n}|v\rangle_{(r)} = 0, \quad \forall\, n \in \mathbb{Z}. \tag{3.83}$$

Following the same procedure with the bosonic part of the vertex, we find for the fermionic part that

$$|E_f\rangle = \exp(E_F^{plane\ wave})|0\rangle, \tag{3.84}$$

where

$$\begin{aligned} E_F^{plane\ wave} &= \frac{1+\Pi}{2}\left[\sum_{r,s=1}^{3}\sum_{m,n=1}^{\infty} b^\dagger_{(r),-m} Q^{rs}_{mn} b^\dagger_{(s),n} - \sqrt{\alpha'}\Lambda \sum_{r=1}^{3}\sum_{m=1}^{\infty} Q^r_m b^\dagger_{(r),-m}\right] \\ &+ \frac{1-\Pi}{2}\left[\sum_{r,s=1}^{3}\sum_{m,n=1}^{\infty} b^\dagger_{(r),m} Q^{rs}_{mn} b^\dagger_{(s),-n} - \frac{\alpha}{\sqrt{\alpha'}}\Theta \sum_{r=1}^{3}\sum_{m=1}^{\infty} Q^r_m b^\dagger_{(r),m}\right] \\ &- \sum_{r=1}^{2}\sqrt{\frac{\alpha_r}{|\alpha_3|}}\, b^\dagger_{(r),0} b^\dagger_{(3),0}, \end{aligned} \tag{3.85}$$

with

$$\Lambda = \alpha_1 \lambda_{(2),0} - \alpha_2 \lambda_{(1),0} \tag{3.86}$$

and

$$\Theta \equiv \frac{1}{\alpha_3}\left(\theta_{(1),0} - \theta_{(2),0}\right). \tag{3.87}$$

The Qs are

$$Q^{rs}_{mn} = \mathrm{sgn}(\alpha_r)\sqrt{\left|\frac{\alpha_s}{\alpha_r}\right|}\left(U^{1/2}_{(r)} C^{1/2} N^{rs} C^{-1/2} U^{1/2}_{(s)}\right)_{mn}, \tag{3.88}$$

$$Q^r_m = \frac{\mathrm{sgn}(\alpha_r)}{\sqrt{|\alpha_r|}}\left(U^{1/2}_{(r)} C^{1/2}_{(r)} C^{1/2} N^r\right)_m. \tag{3.89}$$

The prefactor

Like the flat background case, the necessity of the prefactor comes from the requirement that the supersymmetric algebra closes at first order in the string coupling constant.

Without going deep into the details, we will just present the result. The prefactor is

$$h_3 = -\frac{\alpha'}{\alpha}(1 - 4\mu\alpha K)\left[\frac{1}{4}(\mathcal{K}^2 + \widetilde{\mathcal{K}}^2) + \frac{1+\Pi}{2}\mathcal{W}_\Lambda \mathcal{Y}_\Lambda + \frac{1-\Pi}{2}\mathcal{W}_\Theta \mathcal{Y}_\Theta\right]. \tag{3.90}$$

In the above, it is $K = -(1/4)\, B\Gamma^{-1}B$ and

$$\mathcal{K} = \mathcal{K}_0 + \mathcal{K}_+ + \mathcal{K}_- \,, \qquad \widetilde{\mathcal{K}} = \mathcal{K}_0 + \mathcal{K}_+ - \mathcal{K}_- \,. \tag{3.91}$$

Also, for the fermions, it is

$$\mathcal{K}_0 = \mathbb{P} - i\mu\frac{\alpha}{\alpha'}\mathbb{R} = \sqrt{\frac{2}{\alpha'}}\sqrt{\mu\alpha_1\alpha_2}\big(\sqrt{\alpha_1}a_0^{\dagger(2)} - \sqrt{\alpha_2}a_0^{(1)\dagger}\big)\,, \tag{3.92}$$

$$\mathcal{K}_+ = -\frac{1}{\sqrt{\alpha'}}\frac{\alpha}{1-4\mu\alpha K}\sum_{r=1}^{3}\sum_{n=1}^{\infty}\left[\frac{1}{\alpha_r}(CC_{(r)}^{1/2}U_{(r)}^{-1}N^r)_n\right]a_{n(r)}^{\dagger}\,, \tag{3.93}$$

$$\mathcal{K}_- = -\frac{i}{\sqrt{\alpha'}}\frac{\alpha}{1-4\mu\alpha K}\sum_{r=1}^{3}\sum_{n=1}^{\infty}\left[\frac{1}{\alpha_r}(CC_{(r)}^{1/2}N^r)_n\right]a_{-n(r)}^{\dagger}\,. \tag{3.94}$$

In addition, it is

$$\mathcal{Y}_\Lambda = \frac{1+\Pi}{2}\left[\Lambda - \frac{\alpha}{\sqrt{\alpha'}}\sum_{r=1}^{3}\frac{e(\alpha_r)}{\sqrt{|\alpha_r|}}\frac{(C_{(r)}^{1/2}C^{1/2}U_{(r)}^{-1/2}N^r)_n}{1-4\mu\alpha K}b_{n(r)}^{\dagger}\right]\,, \tag{3.95}$$

$$\mathcal{Y}_\Theta = \frac{1-\Pi}{2}\left[\frac{\alpha}{\alpha'}\Theta + \frac{\alpha}{\sqrt{\alpha'}}\sum_{r=1}^{3}\frac{e(\alpha_r)}{\sqrt{|\alpha_r|}}\frac{(C_{(r)}^{1/2}C^{1/2}U_{(r)}^{-1/2}N^r)_n}{1-4\mu\alpha K}b_{-n(r)}^{\dagger}\right]\,,$$

and

$$\mathcal{W}_\Lambda = -\frac{1+\Pi}{2}\left[\frac{\alpha}{\sqrt{\alpha'}}\sum_{r=1}^{3}\frac{1}{\sqrt{|\alpha_r|^3}}\frac{(C_{(r)}^{1/2}C^{3/2}U_{(r)}^{-1/2}N^r)_n}{1-4\mu\alpha K}b_{-n(r)}^{\dagger}\right]\,, \tag{3.96}$$

$$\mathcal{W}_\Theta = \frac{1-\Pi}{2}\left[\frac{\alpha}{\sqrt{\alpha'}}\sum_{r=1}^{3}\frac{1}{\sqrt{|\alpha_r|^3}}\frac{(C_{(r)}^{1/2}C^{3/2}U_{(r)}^{-1/2}N^r)_n}{1-4\mu\alpha K}b_{n(r)}^{\dagger}\right]\,.$$

Uniqueness of the vertex

The vertex we have presented is based on the true vacuum of the theory (see [86], [87], [88]). The state $|0\rangle$ upon which the vertex is built is the state of minimum energy (zero). This construction has the advantage that it realises the symmetries of the background explicitly. It has the disadvantage that it does not roll to the flat case for $\mu \to 0$.

Originally, the first vertex to appear in the literature ([89], [90], [91], [92] see also [93], [94]) was built on a different state, $|0\rangle'$, defined to be

$$a_n^{\dagger}|0\rangle' = 0, \quad \forall n, \tag{3.97}$$

$$\beta_n^{\dagger}|0\rangle' = 0, \quad n \neq 0, \tag{3.98}$$

$$\theta_0|0\rangle' = 0. \tag{3.99}$$

This vertex has the advantage of rolling back to the flat case for $\mu \to 0$, but it does not realise the full symmetry of the background. In fact it is invariant under the full $SO(8)$ symmetry. What is more, the state $|0\rangle'$ is not the vacuum of the theory, since it has energy 4μ.

The choice of the ground state upon which we will built the 3-string vertex affects only the fermionic part and the prefactor. In both cases, the bosonic part remains the same.

It seems that we have two choices for building the vertex. Either we go for an explicit realisation of the symmetries at the price of smooth flat limit, or we impose the smooth limit condition, but we loose explicit realisation of the symmetries.

What is more, it has been recently argued recently [95], [96] that the correct choice is neither. Instead, the 3-string vertex should be the average of the these two choices. Later, we will argue against the continuous $\mu \to 0$ vertex.

In any case, the quest for the 3-string vertex in the plane wave background is still open. The reader might want also to consult [97], [98], [99], [100], [101].

3.4 Summary

The significance of the plane wave for string theory is twofold. First, it is a background where string theory can be solved and quantised. Furthermore one can proceed and write a string field theory, despite the fact that everything is limited in the light cone gauge. Thus it is worthy studying for its own sake. Secondly, the plane wave arose as a certain Penrose limit of the $AdS_5 \times S^5$ background and therefore, studying string theory in it can shed light in the AdS/CFT correspondence. In fact, there is a conjectured duality proposed already, the BMN correspondence, connecting string theory in the plane wave with a certain sector of $\mathcal{N} = 4$ Super Yang Mills.

What we have done in this chapter is to give a short overview of string theory in the plane wave background, emphasizing on those aspects of the theory that we will need in the subsequent chapters. We demonstrated that string theory can be solved and quantised in the light cone gauge. Based on that fact, we gave an outline of string field theory in the light cone gauge. We demonstrated how one can develop the free theory and we presented a discussion about the simplest vertex one could construct, the 3-string vertex for closed strings. The open string case is similar, practically one sector (either the left or the right moving) of the closed string.

Chapter 4

Causality in Field Theory and String Theory

The main topic of this chapter is the issue of causality in (free) string theory. However, in order to understand it better, we start with classical physics and the Special Theory of Relativity. Then we proceed with point particle quantum field theory and examine the issue of causality in that context. Finally, we ask the same questions for strings.

For a review of special relativity, we refer the reader to [3]. In addition every book on general relativity starts with a presentation of special relativity, see for example [4], [5], [6], [7]. Causality in quantum field theory is discussed in [102], [8], [103], [9]. The string light cone for a flat spacetime using light cone string field theory was first obtained in [58], with further discussion in [59]. Later it was studied and extended to the superstring in [61]. An interesting presentation can also be found in [104], [105], [106]. The discussion for the string light cone in the plane wave background was first presented in [1], where in addition the light cone for a particle in the plane wave was presented, using field theory methods. Independently, in [107], the particle light cone was obtained using geometrical methods.

4.1 Causality in Classical Physics

Causality means that the cause precedes the effect. To use an example, first we pull the trigger of a gun and then the gun fires. If things happened in the reverse order, something really spooky would be going on! In more abstract terms, suppose that we have two events A and B. Event means something that happens at a certain place, at a specific time. Given a coordinate system, let us have for the spacetime coordinates of the events, $A = (t_1, \mathbf{x}_1)$ and $B = (t_2, \mathbf{x}_2)$. Without loss of generality, we can assume that $t_2 > t_1$, that is even A precedes event B (we leave aside for the moment the case that they are simultaneous, $t_1 = t_2$).

If the outcome of event A can *in principle* influence the event B, then we say that the two events are causally related, otherwise they are causally unrelated. For example, suppose that event A is an explosion of a bomb. Fragments of the bomb leave the explosion point $\mathbf{x}_1$ and suppose that a certain piece reaches point $\mathbf{x}_2$ at t_2, where little John is about to buy his new computer game. The fragment knocks him down and John does not buy the game. This means that event A (bomb explosion) influenced event B, the two events are causally related. However, it could be that no fragment reached John (lucky him!) or that he was inside a big building, well protected (so the fragment was blocked). The events are still causally related, because what is important is whether or not there could be an influence in principle. Whether there really was or not, is not the question here.

Based on the above example, we can alter the question and ask whether or not there could be a physical object that could propagate from $(t_1, \mathbf{x}_1)$ to $(t_2, \mathbf{x}_2)$. If the answer is yes, then equivalently we can claim that the two events are causally related, otherwise, they are not.

In Newtonian physics, the answer to that question is always yes. There is absolutely no reason why any object could not travel fast enough and (starting from event A) reach event B. However, Newtonian physics is an approximation. We know that there is an upper bound on the possible speed any object can acquire and thus the answer to our question is "it depends". If the events A and B are such that it would require an superluminal speed for our hypothetical object, then there is no way that one event could influence the other. They are causally unrelated. On the other hand, if the required speed is less than or equal to the speed of light, then they are causally related. The marginal case, where the required speed is exactly the speed of light can be achieved provided that the travelling object is massless.

Given an event, this defines a hypersurface in spacetime that separates the causally related from the causally unrelated events. This surface has the shape of the cone and thus it is call the *light cone*. We see therefore, that the concept of causality changes dramatically in Special Relativity. Given an event, the spacetime is divided into two regions, one consisting of events that are causally related and another consisting of causally unrelated events.

Special Relativity is based on two axioms:

- The laws of physics are the same in every inertial frame of reference.
- The speed of light traveling in the vacuum c is the same is every inertial frame of reference.

This in turn implies that, there is an upper bound on the speed an object can achieve. This upper bound is of course the speed of light c. Nothing can travel faster than light. And if something can travel as fast as light, then it has to be massless.

As a consequence, when we are going from one inertial frame of reference to another, the spatial distance between two points, or the time interval between two events are not invariant. Two inertial observers in relative motion will disagree for the lengths and times. They will not disagree for the spacetime distance

$$ds^2 = -c^2 dt^2 + d\mathbf{x}^2. \tag{4.1}$$

Let us study the motion of a photon. Its velocity is

$$\mathbf{v} = \frac{d\mathbf{x}}{dt} \tag{4.2}$$

and we have for its speed that

$$\mathbf{v}^2 = c^2. \tag{4.3}$$

Combining (4.1), (4.2) and (4.3), we can show that for the photon it is always

$$ds^2 = 0. \tag{4.4}$$

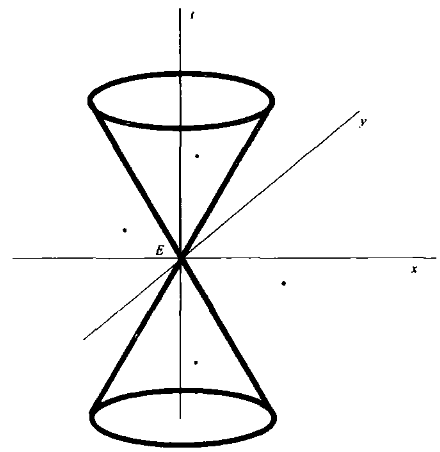

Figure 4.1: The light cone of an event E. The green dots represent events that are causally related to E. The red dots represent causally unrelated events. The blue dots represent the marginal case (they lie on the light cone).

For another particle that is moving slower than light, it is

$$ds^2 < 0, \tag{4.5}$$

while for a (hypothetical) particle that would move faster than a photon it is

$$ds^2 > 0. \tag{4.6}$$

We see therefore that given an event, E, the light cone that corresponds to it is given by the condition

$$ds^2 = 0. \tag{4.7}$$

Events with negative ds^2 fall inside the light cone and are causally related to E. Events with positive ds^2 are outside the light cone and are causally unrelated to E. Schematically, and suppressing some of the spacial dimension, this is shown in figure (4.1).

For finite intervals, the condition for the light cone is

$$\Delta s^2 \equiv c^2 \Delta t^2 - \Delta \mathbf{x}^2 = 0. \tag{4.8}$$

Therefore, in classical physics, given two events, in order to determine if they are causally related or not, all we have to do is examine Δs^2,

$$\Delta s^2 = \begin{cases} > 0, & \text{spacelike - causally unrelated,} \\ = 0, & \text{lightlike - causally related (marginally),} \\ < 0, & \text{timelike - causally related.} \end{cases} \tag{4.9}$$

This condition must hold for arbitrary close points (infinitesimal separations), and for that reason it is also known as *microcausality*. Evidently, if microcausality holds, causality in the greater scheme of things follows.

In anticipation of the same discussion for strings, we write it in light cone coordinates

$$-2\Delta x^+ \Delta x^- + \Delta x^i \Delta x^i \gtreqless 0. \tag{4.10}$$

4.2 Causality in Quantum Mechanics

If the basic laws of nature were classical, then we would be done with causality. However, at the basic and fundamental level, physical laws are quantum mechanical. We have therefore to ask the same questions about causality in the framework of quantum theory.

Naively, we would examine whether a particle can propagate from event A to event B. The amplitude is

$$U(t) = \langle \mathbf{x}_2 | e^{-iHt} | \mathbf{x}_1 \rangle. \tag{4.11}$$

For a freely propagating nonrelativistic particle, it is $H = \mathbf{p}^2/(2m)$. Then the amplitude (4.11) can be computed and it is

$$U(t) = \left(\frac{m}{2\pi i t}\right)^{3/2} e^{im(\mathbf{x}_2 - \mathbf{x}_1)^2/2t}. \tag{4.12}$$

This result is non zero everywhere, which means that the nonrelativistic particle can propagate between any two events. Of course this is in disagreement with the classical result, but should be expected since we are doing nonrelativistic quantum mechanics.

Using instead $H = \sqrt{\mathbf{p}^2 + m^2}$ as the Hamiltonian in (4.11), we get

$$U(t) = \frac{1}{2\pi |\mathbf{x}_2 - \mathbf{x}_1|} \int_0^\infty dp \; e^{-it\sqrt{p^2+m^2}} \sin(p|\mathbf{x}_2 - \mathbf{x}_1|) p. \tag{4.13}$$

The integral can be calculated in terms of Bessel functions. What is important is that this amplitude is small outside the light cone, but still nonzero, which means that causality is violated. Or is it?

We know that relativistic quantum mechanics is plagued with problems of negative probabilities and negative energy states, problems that render the theory unacceptable. The true and consistent merging of quantum mechanics and special relativity comes under the roof of quantum field theory. It is within this framework that we should examine the issue of causality.

In quantum field theory, the basic object is the field itself. So, we should examine under which conditions a particle is allowed to propagate from A to B. But... wait a minute. Quantum mechanical particles behave quite differently than small pellets. They can do all sorts of strange and unphysical things, as long as they can go undetected. It is the uncertainty principle that allows them to do so. So the question in terms of particle propagation is not the right one. Causality will not be violated if a particle travels faster than the speed of light undetected. Causality will be violated if measurements at spacelike separated points A and B are allowed to interfere.

For the two measurements of a physical quantity, corresponding to the operator $\mathcal{O}(x)$, not to interfere, we must have

$$[\mathcal{O}(x_1), \mathcal{O}(x_2)] = 0, \tag{4.14}$$

for $(x_1 - x_2)^2 > 0$. Every observable of the theory is built by the field and its derivatives. Without loss of generality, we can restrict ourselves to the real Klein-Gordon field. The question whether the theory is causal or not translates into when the commutator $[\phi(x_1), \phi(x_2)]$ is zero identically.

The Klein-Gordon field is the first field studied in every book of quantum field theory. The commutator under question is

$$[\phi(x_1), \phi(x_2)] = \Delta_+(x_1 - x_2) - \Delta_+(x_2 - x_1), \tag{4.15}$$

where we have defined the integral

$$\Delta_+(x) \equiv \int \frac{d\mathbf{p}}{(2\pi)^3} \frac{1}{2p^0} e^{ip\cdot x}. \tag{4.16}$$

For spacelike separated events, we can go to a frame of reference where $x^0 = 0$ and $|\mathbf{x}| = \sqrt{x^2}$. Then (4.16) becomes

$$\Delta_+(x) = \frac{m}{4\pi^2\sqrt{x^2}} \int_0^\infty \frac{udu}{u^2+1} \sin(m\sqrt{x^2}u), \tag{4.17}$$

where we also have performed a change in the integration variable, $u \equiv p/m$. This can be evaluated in terms of the Hankel function and it is

$$\Delta_+(x) = \frac{m}{4\pi^2\sqrt{x^2}} K_1(m\sqrt{x^2}). \tag{4.18}$$

Notice now that (4.18) for $x^2 > 0$ is even, so the two terms in (4.15) cancel each other. Thus, for spacelike separated events the commutator is zero. Otherwise such cancelation could not occur (for x^2 timelike, the Lorentz transformation we performed above can not be done) and the result is nonzero.

The generalisation to other fields (like the Dirac or the electromagnetic) is straight forward, with the understanding that for fermionic fields we should examine the anticommutator. If fact, demanding causality in accordance with special relativity in a quantum field theory forces us to quantise integral spin fields with commutators and half integral spin fields with anticommutators. Causality and relativity in quantum theory are behind the spin-statistics theorem.

In quantum mechanics one could use a different approach and discuss causality in the context of S-matrix. More specifically, the question would be which conditions are needed to ensure that the S-matrix is Lorentz invariant. This would correspond to macroscopic causality in classical physics. One can show ([9]) that microscopic causality and locality of the theory imply macroscopic causality. This holds the other way around as well. But we believe that microscopic causality is more fundamental than macroscopic. After all, there are situations where an S-matrix is not well defined because asymptotic states are not well defined, yet one can still define interaction vertices and, most important for our discussion, microcausality can still be defined as above.

It will be interesting to use this method to obtain the light cone for in a different background[1]. In anticipation of the same calculation for string theory, we will obtain

[1]The following calculation is based on [1].

the particle light cone in the plane wave (3.7). For simplicity, we will use a real scalar field $\phi(x)$, of mass M. The equation of motion for the massive Klein-Gordon field is,

$$\frac{1}{\sqrt{|g|}}\partial_\mu\left(\sqrt{|g|}g^{\mu\nu}\partial_\nu\right)\phi - M^2\phi = 0 \Rightarrow$$
$$-2\partial_+\partial_-\phi + \mu^2(x^i)^2\partial_-^2\phi + \partial_i\partial_i\phi - M^2\phi = 0. \tag{4.19}$$

By Fourier transforming x^-,

$$\phi(x^+, x^-, \vec{x}) = \int \frac{dp^+}{2\pi} e^{-ix^-p^+}\varphi(x^+, p^+, \vec{p}), \tag{4.20}$$

we have

$$i\frac{\partial\varphi}{\partial x^+} = H\varphi, \tag{4.21}$$

where

$$H = \frac{1}{2p^+}\left(-\frac{\partial^2}{\partial x^{i2}} + M^2 + m^2(x^i)^2\right). \tag{4.22}$$

We have defined $m \equiv \mu p^+$. It is easy to see that the solution for the field is

$$\phi(x) = \int \frac{dp^+}{2\pi} e^{-ix^+p^- - ix^-p^+} \sum_{\{n^i\}} a\left(p^+, \{n^i\}\right) \prod_{i=1}^{D-2} C(n^i)H_{n^i}(\sqrt{m}x^i)e^{-m(x^i)^2/2} + h.c. \tag{4.23}$$

H_n are Hermite polynomials, $C(n^i)$ is their normalisation,

$$C(n^i) = \sqrt{\frac{\sqrt{m}}{2^{n^i}(n^i!)\sqrt{\pi}}}. \tag{4.24}$$

For the energy we have that

$$p^- = E = \frac{1}{2p^+}\left[M^2 + 2m\sum_{i=1}^{D-2}\left(n^i + \frac{1}{2}\right)\right]. \tag{4.25}$$

We can quantise the field by promoting it to an operator and imposing the equal time commutation relation

$$\left[\phi(x^+, x^-, \vec{x}), \phi(x^+, y^-, \vec{y})\right] = \delta(x^- - y^-)\prod_{i=1}^{D-2}\delta(x^i - y^i) \tag{4.26}$$

which in terms of the field modes $a, a^\dagger$ reads

$$\left[a(p^+, \{n^i\}), a^\dagger(q^+, \{m^j\})\right] = 2\pi\delta(p^+ - q^+)\delta_{\{n^i\},\{m^j\}}. \tag{4.27}$$

Clearly, $a, a^\dagger$ are annihilation and creation operators, respectively.

The calculation of the commutator of two fields is straight forward to perform and one finds that

$$[\phi(x), \phi(y)] = \tilde{g}_1 - \tilde{g}_2, \tag{4.28}$$

where

$$\begin{aligned} \tilde{g}_1 &= \int \frac{dp^+}{2\pi} e^{-i\Delta x^- p^+} e^{-i\Delta x^+ \frac{M^2}{2p^+}} \prod_{i=1}^{D-2} \sqrt{\frac{\mu p^+}{\pi}} \sqrt{\frac{1}{2i \sin(\Delta x^+ \mu)}} \\ & \exp\left\{ \frac{\mu p^+}{2i \sin(\mu \Delta x^+)} \left(2x^i y^i - \left((x^i)^2 + (y^i)^2\right) \cos(\mu \Delta x^+)\right) \right\}. \end{aligned} \tag{4.29}$$

As usual, we have abbreviated $\Delta x^+ \equiv x^+ - y^+$. $\tilde{g}_2$ is obtained form $\tilde{g}_1$ by interchanging x with y.

Notice that in the limit $\mu \longrightarrow 0$, (4.29) has a good limit and in fact reduces to the propagator of the free particle. This is something we should expect, since (4.29) is essentially, the Green function of the simple harmonic oscillator with μ being the frequency and the Green function for the harmonic oscillator for zero frequency becomes the Green function of the free particle. On the other hand, for $\mu \longrightarrow 0$, the plane wave metric reduces to the flat metric and it is (at least conceptually) clear that in that case we would essentially be solving for the Green function of the freely propagating particle. It is nice to see how different things fit together.

In order to find the light cone, we seek the condition under which the commutator (4.28) vanishes identically. For that purpose, we perform an analytic continuation on $\tilde{g}_1$,

$$p^+ \longrightarrow ip^+ \tag{4.30}$$

and we have that

$$\begin{aligned} \tilde{g}_1 \longrightarrow \tilde{g} &= \int \frac{dp^+}{2\pi} i e^{\Delta x^- p^+} e^{-\Delta x^+ \frac{M^2}{2p^+}} \prod_{i=1}^{\infty} \sqrt{\frac{ip^+ \mu}{\pi}} \sqrt{\frac{1}{2 \sin(\Delta x^+ \mu)}} \\ & \exp\left\{ \frac{\mu p^+}{2 \sin(\mu \Delta x^+)} \left(2x^i y^i - \left((x^i)^2 + (y^i)^2\right) \cos(\mu \Delta x^+)\right) \right\}. \end{aligned} \tag{4.31}$$

For the second term in the right hand side of (4.28), we perform the analytic continuation $p^+ \longrightarrow -ip^+$ and we find that $\tilde{g}_2 \longrightarrow \tilde{g}$.

If the integral $\tilde{g}$ converges, the two terms cancel and the commutator is zero. $\tilde{g}$ is a well behaving integral for $p^+ \longrightarrow 0$, but problems of converges arise for $p^+ \longrightarrow \infty$.

The integral will converge only if

$$-\Delta x^- + \frac{\mu}{2\sin(\mu\Delta x^+)} \sum_{i=1}^{D-2} \left[\left((x^i)^2 + (y^i)^2\right) \cos(\mu\Delta x^+) - 2x^i y^i \right] > 0, \tag{4.32}$$

in which case the commutator (4.28) will be zero identically. We conclude therefore that the light cone for a particle in the plane wave (3.7) is

$$L.C._{planewave}^{particle} \equiv -\Delta x^- + \frac{\mu}{2\sin(\mu\Delta x^+)} \sum_{i=1}^{D-2} \left[\left((x^i)^2 + (y^i)^2\right) \cos(\mu\Delta x^+) - 2x^i y^i \right] = 0. \tag{4.33}$$

The condition for causally related/unrelated particles in the plane wave is

$$L.C._{planewave}^{particle} = \begin{cases} > 0, & \text{causally unrelated,} \\ = 0, & \text{causally related, marginal case,} \\ < 0, & \text{causally related.} \end{cases} \tag{4.34}$$

This agrees with the result of [107], where the light cone for a particle in the plane wave was obtained using geometrical methods. For further discussion of the light cone and causality in the plane wave metric we refer the reader to [108], [109]. Notice that the light cone exhibits a periodicity in the light cone time x^+,

$$x^+ \longrightarrow x^+ + 2\pi/\mu. \tag{4.35}$$

What is not obvious is the following. There exists a coordinate transformation that makes the plane wave metric (3.7) conformally flat [110]. Specifically, first we pass the spherical polar coordinates for the transverse part of the plane wave metric and then we change coordinates

$$u = \tan(\mu x^+), \tag{4.36}$$

$$x = x' \cos(\mu x^+), \tag{4.37}$$

$$x^- = x^{-'} - \frac{1}{2}\mu u x^2. \tag{4.38}$$

Then, the plane wave metric becomes the flat metric, times an overall conformal factor. It is remarkable that under this coordinate transformation, the light cone (4.33) becomes the flat, Minkowski light cone.

4.3 Causality in String Theory

We come now to investigate causality in string theory. And we run immediately into the first difficulty. Strings are extended objects, so what is the meaning of the light cone? Do we take as such the light cone of the centre of mass, the middle point (which can not even be defined for closed strings), some other combination? Puzzling.

To overcome this difficulty, we are going to examine causality in the same way we did for point particle field theory. Observables in string theory should be built from the string field and its derivatives. Again, because the theory is a quantum one, the real question is when separate observations are allowed to interfere. As in the point particle case, this reduces to when the commutator of two string fields is allowed to vanish. Unlike point particles, we lack a full, covariant and gauge invariant field theory for strings. However, as we saw in previous chapters, we can write down a string field theory in the light cone gauge. This is the reason that we study string causality using light cone string field theory.

4.3.1 Causality in the flat background

We want to find the condition under which the commutator of two string fields, corresponding to strings $\{x^+, x_0^-, \vec{X}(\sigma)\}$ and $\{y^+, y_0^-, \vec{Y}(\sigma)\}$ vanishes identically. Consider for simplicity the case of two open bosonic strings. We have computed the commutator of their fields in chapter 2, equations (2.25), (2.27), which we repeat here for convenience.

$$\left[\Phi(x^+, x_0^-, \vec{X}), \Phi(y^+, y_0^-, \vec{Y})\right] = G_{open}^{bosonic}(X;Y) - \{x \leftrightarrow y\}, \tag{4.39}$$

$$\begin{aligned} G_{open}^{bosonic}(X;Y) &= \int_0^\infty \frac{dp^+}{2\pi}\left(-\frac{ip^+}{2\pi\Delta x^+}\right)^{(D-2)/2} \exp\left(\frac{i\Delta\vec{x}_0{}^2 p^+}{2\Delta x^+}\right) e^{-i\Delta x^- p^+} \\ &\quad \prod_{l=1}^{\infty}\prod_{i=1}^{D-2}\sqrt{\frac{l}{2\pi\alpha'}}\sqrt{\frac{1}{2i\sin\frac{l\Delta x^+}{2\alpha' p^+}}} \\ &\quad \exp\left\{\frac{l/(2\alpha')}{2i\sin\frac{l\Delta x^+}{2\alpha' p^+}}\left(2x_l^i y_l^i - \left((x_l^i)^2 + (y_l^i)^2\right)\cos\frac{l\Delta x^+}{2\alpha' p^+}\right)\right\}, \end{aligned} \tag{4.40}$$

with the abbreviations $\Delta x^+ \equiv x^+ - y^+$, $\Delta x^- \equiv x_0^- - y_0^-$, $\Delta\vec{x}_0 \equiv \vec{x}_0 - \vec{y}_0$.

Let us focus on the first term in the right hand side of (4.39) assuming without loss of generality that $\Delta x^+ > 0$. We will perform a contour deformation, by sending

$$p^+ \longrightarrow ip^+. \tag{4.41}$$

Then it is

$$\begin{aligned} G \; = \; & \int_0^\infty \frac{dp^+}{2\pi} i \left(\frac{p^+}{2\pi \Delta x^+}\right)^{(D-2)/2} \exp\left[\left(-\frac{\Delta \vec{x}_0^{\,2}}{2\Delta x^+} + \Delta x^-\right) p^+\right] \\ & \prod_{l=1}^{\infty} \prod_{i=1}^{D-2} \sqrt{\frac{l}{2\pi\alpha'}} \sqrt{\frac{1}{2\sinh \frac{l\Delta x^+}{2\alpha' p^+}}} \\ & \exp\left\{ \frac{l/2\alpha'}{2\sinh\left(\frac{l\Delta x^+}{2\alpha' p^+}\right)} \left[2x_l^i y_l^i - \left((x_l^i)^2 + (y_l^i)^2\right)\cosh \frac{l\Delta x^+}{2\alpha' p^+}\right]\right\}. \end{aligned} \tag{4.42}$$

We can manipulate the second term on the right hand side of (4.39), in a similar manner, sending $p^+ \longrightarrow -ip^+$. The result is equal to G. If the integral converges, the two terms cancel each other and the commutator is zero.

The integral G is converging for $p^+ \to 0$. For $p^+ \to \infty$, it will converge only if

$$L.C._{flat}^{open\ string} \equiv -2\Delta x^+ \Delta x_0^- + \Delta \vec{x}_0^2 + \sum_{l=1}^{\infty} \Delta \vec{x}_l^2 > 0. \tag{4.43}$$

If this condition is not satisfied, the integrals do not converge, we are not permitted to do the analytic continuation and of course the commutator is not zero. The condition therefore that determines when two strings are causally related or not is

$$L.C._{flat}^{open\ string} = \begin{cases} > 0, & \text{causally unrelated,} \\ = 0, & \text{causally related, marginally,} \\ < 0, & \text{causally related.} \end{cases} \tag{4.44}$$

The string light cone for open strings is *defined* to be the hypersurface in space-time that satisfies

$$-2\Delta x^+ \Delta x_0^- + \Delta \vec{x}_0^{\,2} + \sum_{l=1}^{\infty} \Delta \vec{x}_l^2 = 0. \tag{4.45}$$

For closed strings, the calculation is exactly the same and the answer is

$$-2\Delta x^+ \Delta x_0^- + \Delta \vec{x}_0^{\,2} + \sum_{l=1}^{\infty} \sum_{i=1}^{\infty} \left((\Delta x_l^i)^2 + (\Delta \tilde{x}_l^i)^2\right) = 0. \tag{4.46}$$

Analysing the propagator (2.108) in the same way, the condition that one recovers is

$$L.C._{flat}^{closed\ string} = \begin{cases} > 0, & \text{causally unrelated,} \\ = 0, & \text{causally related, marginally,} \\ < 0, & \text{causally related.} \end{cases} \tag{4.47}$$

where of course now we have that

$$L.C._{flat}^{closed\ string} \equiv -2\Delta x^+ \Delta x_0^- + \Delta \vec{x}_0^{\,2} + \sum_{l=1}^{\infty}\sum_{i=1}^{\infty} \left((\Delta x_l^i)^2 + (\Delta \tilde{x}_l^i)^2\right). \tag{4.48}$$

We see immediately that the string light cone is different than the point particle light cone. This is something we should expect, since the string is a spatially extended object. Another thing to notice is that the modification to the light cone as we know it from point particles comes exclusively from the internal oscillating modes of the string. In addition, the zero mode of the string, which corresponds to the centre of mass behaves exactly like a point particle. The string light cone (4.45) truncated to the zero mode is the same with the point particle light cone (4.8). Finally, the string light cone, like the point particle light cone respects the symmetries of the background, in this case emphasis on the translational invariance. We will comment further on this at chapter 6.

4.3.2 Causality in the plane wave background

We saw that the string light cone for strings propagating in a flat background is quite different than the point particle light cone. It is only natural to want to examine the situation for the plane wave background. As before, we will carry the analysis for the open string, focusing on the bosonic part.

The propagator is, as we found in (3.43),

$$\begin{aligned} \tilde{G}_{open}^{bosonic} &= \int_0^{\infty} \frac{dp^+}{2\pi} e^{-i\Delta x_0^- p^+} \prod_{l=0}^{\infty}\prod_{i=1}^{8} \sqrt{\frac{\omega_l}{2\pi\alpha'}} \sqrt{\frac{1}{2i \sin \frac{\omega_l \Delta x^+}{2\alpha' p^+}}} \\ &\exp\left\{ \frac{\omega_l/(2\alpha')}{2i\sin\left(\frac{\Delta x^+ \omega_l}{2\alpha' p^+}\right)} \left[2x_l^i y_l^i - \left((x_l^i)^2 + (y_l^i)^2\right) \cos\left(\frac{\Delta x^+ \omega_l}{2\alpha' p^+}\right)\right]\right\} \end{aligned} \tag{4.49}$$

Without loss of generality we can take $\Delta x^+ > 0$ and perform the analytic continuation

$$p^+ \longrightarrow ip^+. \tag{4.50}$$

Then, the propagator takes the form

$$\begin{aligned}\tilde{G} &= \int_0^\infty \frac{dp}{2\pi} ie^{\Delta x_0^- p^+} \prod_{l=0}^{\infty}\prod_{i=1}^{8} \sqrt{\frac{\tilde{\omega}_l}{2\pi\alpha'}} \sqrt{\frac{1}{2\sinh\frac{\Delta x^+ \tilde{\omega}_l}{2\alpha' p^+}}} \\ &\exp\left\{ \frac{\tilde{\omega}_l/2\alpha'}{2\sinh\frac{\Delta x^+\tilde{\omega}_l}{2\alpha' p^+}} \left(2x_l^i y_l^i - \left((x_l^i)^2 + (y_l^i)^2\right)\cos\frac{\Delta x^+\tilde{\omega}_l}{2\alpha' p^+}\right)\right\}. \end{aligned} \tag{4.51}$$

Remember that there is an implicit p^+ dependence through ω_l. After the analytic continuation we have

$$\omega_l \longrightarrow \sqrt{l^2 - (2\alpha' p^+ \mu)^2} \equiv \tilde{\omega}_l. \tag{4.52}$$

For $p^+ \to 0$, the integral converges. For $p^+ \to \infty$, the integral converges only if

$$-\Delta x_0^- + \frac{\mu}{2\sin(\Delta x^+ \mu)} \sum_{l=0}^{\infty}\sum_{i=1}^{8} \left[\left((x_l^i)^2 + (y_l^i)^2\right)\cos(\Delta x^+\mu) - 2x_l^i y_l^i\right] > 0. \tag{4.53}$$

In obtaining this result, one should be careful with $\tilde{\omega}$, since, for each fixed l in the $p^+ \to \infty$ it becomes imaginary, $\tilde{\omega}_l \to 2i\alpha' p^+\mu$. The second term in the right hand side of the string field commutator can be shown to be equal to the first, if one make the analytic continuation $p^+ \longrightarrow -ip^+$. Then, if (4.53) holds, the two integrals cancel and the commutator is zero. Otherwise, we can not cancel them and the commutator of course has a non-zero value.

We conclude that the condition for the string light cone in the plane wave background is

$$L.C.^{open\ string}_{planewave} = \begin{cases} > 0, & \text{causally unrelated,} \\ = 0, & \text{causally related, marginally,} \\ < 0, & \text{causally related,} \end{cases} \tag{4.54}$$

where we have defined

$$L.C.^{open\ string}_{planewave} \equiv -\Delta x_0^- + \frac{\mu}{2\sin(\Delta x^+\mu)} \sum_{l=0}^{\infty}\sum_{i=1}^{8} \left[\left((x_l^i)^2 + (y_l^i)^2\right)\cos(\Delta x^+\mu) - 2x_l^i y_l^i\right]. \tag{4.55}$$

We define the string light cone in the plane wave background to be the hypersurface that obeys

$$-\Delta x_0^- + \frac{\mu}{2\sin(\Delta x^+\mu)} \sum_{l=0}^{\infty}\sum_{i=1}^{8} \left[\left((x_l^i)^2 + (y_l^i)^2\right)\cos(\Delta x^+\mu) - 2x_l^i y_l^i\right] = 0. \qquad (4.56)$$

We see immediately that the string light cone is different than the underlying point particle one. The modification again comes exclusively form the internal oscillating modes of the string. In fact, if we truncate (4.56) to the zero modes, we recover the point particle light cone for the plane wave background, (4.33). In accordance with the plane wave metric, (3.7), translational invariance is lost now. Finally, we experience the same periodicity in x^+, as we did with (4.56), despite the fact that we have not imposed any such condition. We interpret this result as a requirement for the consistency of the theory. We will say more on this at chapter 6.

4.4 Summary

In this chapter we have discussed the issue of microscopic causality in string theory. We started by explaining what causality is and how it works in classical physics, introducing the notion of the light cone. Then we examined it in the context of point particle field theory, where we found out that the point particle light cone remains the same as in the classical physics. Based on this knowledge, we examined causality in string theory, deploying the machinery of light cone string field theory. There we found that the string light cone is quite different from the point particle one. The internal oscillating modes of the string modify its shape, although the zero mode of the string (which corresponds to the centre of mass) behaves exactly as a point particle. Carrying over the analysis to the plane wave background, we found that the same stringy feature of the sting light cone remain. Even more, we discovered that the plane wave background has to have a periodicity in x^+, to make the theory consistent.

Chapter 5

Causality Revisited - Interactions

In ordinary quantum field theory, interaction do not modify the causal structure of the theory. This is achieved by the requirement of locality, that is that interactions should be local in the fields and their derivatives. For example, for the real Klein-Gordon field, the simplest interaction we could have is a ϕ^3 interaction. This is implemented in the Hamiltonian of the theory by the term

$$H_{int} = g \int d^3x \phi^3(x). \tag{5.1}$$

Notice that all three fields in the interaction are taken at the same spacetime point x. This is the requirement of locality.

We will examine now what happens in string theory.

5.1 The flat background case

The first indication that string interactions modify the string light cone came in [61]. There the authors calculated a certain amplitude, namely

$$\mathcal{A} = \langle 0|\Phi[p_3^+, \vec{X}_{(3)}(\sigma)] \left[\dot{\Phi}[x^-_{(1),0}, \vec{X}_{(1)}(\sigma)], \Phi[x^-_{(2),0}, \vec{X}_{(2)}(\sigma)]\right] |0\rangle, . \tag{5.2}$$

The dot in (5.2) stands for the time derivative and all fields are taken to be at $x^+ = 0$. $\dot{\Phi}$ can be replaced by the Heisenberg equation of motion,

$$\dot{\Phi} = i\,[H, \Phi]\,. \tag{5.3}$$

The full Hamiltonian of the theory is

$$H = H_{free} + H_3, \tag{5.4}$$

where H_{free} is of course the Hamiltonian of the free theory and H_3 is the interacting part of the Hamiltonian for a 3 string interactions (see (2.35)). Clearly, the only modification (if any) will come from the interacting term. Therefore the calculation reduces to finding the amplitude

$$\mathcal{A} = i\langle 0|\Phi[p_3^+, \vec{X}_{(3)}(\sigma)] \left[\left[H_3, \Phi[x^-_{(1),0}, \vec{X}_{(1)}(\sigma)]\right], \Phi[x^-_{(2),0}, \vec{X}_{(2)}(\sigma)]\right] |0\rangle. \tag{5.5}$$

It was found that the amplitude does not vanish outside the string light cone. This means that the causal structure of the theory is modified by the interactions. Based on that result, the authors of [62] proceeded to argue that the apparent violation of string causality is more general and might have implications for black hole physics. Let us repeat their calculation here, stating that same calculation for the Klein-Gordon field with a ϕ^3 interaction gives zero outside the particle light cone. Also, the amplitude (5.2) in the plane wave background, vanishes outside the string light cone, as shown in appendix .

Consider the amplitude

$$M = {}_H\langle 0| \left[\Phi_H(1), \Phi_H(2)\right] |3\rangle_H, \tag{5.6}$$

of two string fields, denoted by 1 and 2, with a 3$^{\text{rd}}$ spectator state. The spectator state is necessary for a possible non-zero contribution at first order in the string coupling constant g. The subscript H means that everything is in the Heisenberg picture. Passing to the interaction picture, we have

$$M = \langle 0; x_1^+|\Phi_I(1)U_I(x_1^+, x_2^+)\Phi_I(2)|3; x_2^+\rangle - \{1 \leftrightarrow 2\}, \tag{5.7}$$

where $U_I(x_1^+, x_2^+)$ is the time evolution operator in the interaction picture. In the leading order of string coupling, it is

$$U_I(x_1^+, x_2^+) = 1 - ig\int_{x_1^+}^{x_2^+} dx^+ H_3(x^+) + \cdots . \tag{5.8}$$

Hence up to first order in g we have

$$M = M^{(0)} + M^{(1)}, \tag{5.9}$$

where the zeroth order amplitude

$$M^{(0)} = \langle 0|\left[\Phi_I(1), \Phi_I(2)\right]|3\rangle \tag{5.10}$$

is a matrix element of the commutator of the two string fields, and

$$\begin{aligned} M^{(1)} &= ig\int_{x_1^+}^{x_2^+} dx^+ \langle 0|\Phi_I(1)H_3(x^+)\Phi_I(2) \\ &\quad +\Phi_I(2)H_3(x^+)\Phi_I(1) - \Phi_I(1)\Phi_I(2)H_3(x^+)|3\rangle. \end{aligned} \tag{5.11}$$

For strings outside the string light cone (4.45), we see immediately that $M^{(0)} = 0$. Any possible causality violations will come from a non-zero $M^{(1)}$.

Now, since H_3 is of the form Φ^3 and the string field of the form $\Phi \sim A + A^\dagger$, we can break the interaction vertex down to terms with equal number of creation and annihilation operators, $H_3 = H_{3\mathrm{aaa}} + H_{3\mathrm{aac}} + H_{3\mathrm{acc}} + H_{3\mathrm{ccc}}$. It is easy to see that unless the spectator state is a single string state of the form $|3\rangle = A^\dagger(p_3^+, \{\vec{n}_{3,l}\})|0\rangle$, $M^{(1)}$ will be identically zero. With this choice for the spectator state, we have

$$\begin{aligned} M^{(1)} &= ig\int_{x_1^+}^{x_2^+} dx^+ \langle 0|\Phi_\mathrm{a}(1)H_{3\mathrm{aac}}(x^+)\Phi_\mathrm{c}(2) \\ &\quad +\Phi_\mathrm{a}(2)H_{3\mathrm{aac}}(x^+)\Phi_\mathrm{c}(1) - \Phi_\mathrm{a}(1)\Phi_\mathrm{a}(2)H_{3\mathrm{acc}}(x^+)|3\rangle. \end{aligned} \tag{5.12}$$

Φ_a is the part of the string field containing the annihilation operator A, Φ_c the part with the creation operator $A^\dagger$.

A lengthy computation can show that

$$\begin{aligned} M^{(1)} &= ig\int_{x_1^+}^{x_2^+} dx^+ \left(\prod_{r=1}^{2}\int \frac{d\vec{p}_{(r),0}}{(2\pi)^{D-2}}\int_{-\infty}^{\infty}\frac{dp_r^+}{2\pi}\sum_{\{n^i_{(r),l}\}} f_{\{n^i_{(r),l}\}}(x^i_{(r),l})\right. \\ &\quad \left.\left(e^{-ix^+\sum_{r=1}^{3}p_r^-}e^{i\sum_{r=1}^{2}\left(x_r^+p_r^-+x_r^-p_r^+\right)}\left(\tilde{V}(1,2,3)+\tilde{V}(2,1,3)\right)\right)\right. \\ &\quad \delta\left(\sum_{r=1}^{3}\alpha_r\right)\delta\left(\sum_{r=1}\vec{p}_{(r),0}\right)F(\alpha_1,\alpha_2). \end{aligned} \tag{5.13}$$

The function $F(\alpha_1, \alpha_2)$ is

$$F(\alpha_1,\alpha_2) = \vartheta(\alpha_1)\vartheta(-\alpha_2) + \vartheta(-\alpha_1)\vartheta(\alpha_2) - \vartheta(-\alpha_1)\vartheta(-\alpha_2). \tag{5.14}$$

$\tilde{V}$ is the 3-string vertex in the momentum representation and in the oscillator basis.

To make things more transparent, it will be sufficient to specialize to the case of a tachyon component field and a tachyon spectator state. Then, the 3-string vertex is simply

$$\tilde{V}(1,2,3) = \exp\left(\frac{\tau_0}{2}\sum_{r=1}^{3} p_r^-\right), \tag{5.15}$$

where of course the p_r^- are given by (2.18). In retrospect, the amplitude simplifies to

$$\begin{aligned} M^{(1)} &= 2ig\int_{x_1^+}^{x_2^+} dx^+ \left(\prod_{r=1}^{2}\int \frac{d\vec{p}_{(r),0}}{(2\pi)^{D-2}}\int_{-\infty}^{\infty}\frac{dp_r^+}{2\pi}\right) \\ &\quad \left(e^{-ix^+\sum_{r=1}^{3}p_r^-}e^{i\sum_{r=1}^{2}\left(x_r^+p_r^-+x_r^-p_r^+\right)}\exp\left(\frac{\tau_0}{2}\sum_{r=1}^{3}p_r^-\right)\right) \\ &\quad \delta\left(\sum_{r=1}^{3}\alpha_r\right)\delta\left(\sum_{r=1}\vec{p}_{(r),0}\right)F(\alpha_1,\alpha_2). \end{aligned} \tag{5.16}$$

If the vertex (5.15) was polynomial in α_r, we could show that the amplitude (5.16) vanishes for $x_1 - x_2$ spacelike by the usual contour deformation argument. However this is not the case and we see that the amplitude receives contribution even for spacelike separations.

The reader might be worried here that this is something we should expect since we decided to work with the tachyon. Tachyons are notorious for causing all sorts of causality problems, after all. But this is not the case here. The tachyon is treated as a massive scalar particle, its (negative) mass-squared does not interfere with the calculation. The same phenomenon would appear if its mass was positive (an ordinary massive scalar particle) or zero. In fact, the same conclusion holds for the superstring as well, where the ground state of the string has zero mass.

5.2 The plane wave case

It is only natural to repeat the calculation for the plane wave string theory, to see what are the similarities and differences between the flat and the plane wave theory. Surprisingly, we find that in the plane wave there is no modification to the causal structure of the theory due to interactions.

The first indication came through the computation of the amplitude (5.2). It was found that unlike the free case, by deploying the usual analytic continuation argument, the amplitude vanishes for string configurations outside the string light cone. But of course the real test comes form examining the amplitude (5.6), which after all is more general.

In repeating the calculation for the plane wave, notice that the steps that lead of (5.13) are the same. We have therefore that

$$\begin{aligned} M^{(1)} &= ig\int_{x_1^+}^{x_2^+} d\tau \prod_{r=1}^{2}\int_{-\infty}^{\infty}\frac{d\alpha_r}{\sqrt{2\pi|\alpha_r|}}F(\alpha_1,\alpha_2) \\ &\cdot \sum_{\{\vec{n}_{1,l}\},\{\vec{n}_{2,l}\}}\left(f_{\{\vec{n}_{1,l}\}}(\vec{x}_{1,l})f_{\{\vec{n}_{2,l}\}}(\vec{x}_{2,l})\tilde{V}(1,2,3)\, e^{-i\tau\sum_{r=1}^{3}p_r^-}e^{i\sum_{r=1}^{2}p_r^- x_r^+ + p_r^+ x_r^-}\right) \quad . \end{aligned} \tag{5.17}$$

The function $F(\alpha_1, \alpha_2)$ is the same as before, (5.14). Now of course $\tilde{V}$ is the 3-string vertex in the momentum representation and in the oscillator basis for the plane wave.

It will be convenient to take the kinematical situation to be $\alpha_1, \alpha_3 > 0$ and $\alpha_2 < 0$. Then the interaction term in the Hamiltonian is

$$H_3 = g\int\prod_{r=1}^{3}d\alpha_r\mathcal{D}\vec{P}_{(r)}(\sigma)\tilde{h}(\alpha_r,\vec{P}_{(r)}(\sigma))\Phi[x^+,\alpha_r,\vec{P}_{(r)}(\sigma)], \tag{5.18}$$

where $\tilde{h}$ is the measure and is

$$\begin{aligned} \tilde{h}(\alpha_r,\vec{P}_{(r)}(\sigma)) &= \delta\left(\sum_{r=1}^{3}\alpha_r\right)\int\prod_{r=1}^{3}\mathcal{D}\vec{Y}_{(r)}(\sigma)e^{i\int d\sigma'\vec{Y}_{(r)}(\sigma')\cdot\vec{P}_{(r)}(\sigma')} \\ & \quad \delta\left[\vec{Y}_{(2)}(\sigma)-\vec{Y}_{(3)}(\sigma)-\vec{Y}_{(1)}(\sigma)\right]. \end{aligned} \tag{5.19}$$

By inserting the expansion of the string field in terms of creation/annihilation operators into (5.18), we obtain

$$H_3 = g\int\prod_{r=1}^{3}\frac{d\alpha_r}{2\pi}\sum_{\{n^i_{(r),l}\}}\tilde{V}(\alpha_r,\{n^i_{(r),l}\})\prod_{r=1}^{3}A\left(p_r^+,\{n^i_{(r),l}\}\right)+\ldots\ , \tag{5.20}$$

where the ... are terms of the form $AAA^\dagger, AA^\dagger A^\dagger, A^\dagger A^\dagger A^\dagger$. For our case of interest, only the AAA term is relevant. $\tilde{V}$ is

$$\tilde{V}(a_r,\{n^i_{(r),l}\}) = \int\prod_{r=1}^{3}\prod_{l=0}^{\infty}d\vec{p}_{(r),l}\tilde{h}(\alpha_r,\vec{P}_{(r)}(\sigma))\tilde{f}_{\{\vec{n}_{(r),l}\}}(\vec{p}_{(r),l}), \tag{5.21}$$

where

$$\tilde{f}_{\{\vec{n}_{(r)},l\}}(\vec{p}_{(r)},l) = \int \prod_{l=0}^{\infty} e^{-i\sum_{l=0}^{\infty} \vec{y}_l \cdot \vec{p}_l} f_{\{\vec{n}_{(r)},l\}}(\vec{x}_{(r)},l) \tag{5.22}$$

are the harmonic oscillator eigenfunctions in the momentum space (see also (3.36)).

After this brief digression, back to the calculation of (5.18). The explicit expression of $\tilde{V}(1,2,3)$, is (5.21). Using the sum rule for the Hermite polynomials (2.24), we can easily calculate the sum in the second line of (5.18) and obtain

$$\begin{aligned} M^{(1)} &= ig \int_{x_1^+}^{x_2^+} d\tau \int_{-\infty}^{\infty} \frac{d\alpha_1}{2\pi} \\ &\cdot \int \prod_{r=1}^{3} \prod_{l=0}^{\infty} d\vec{y}_{r,l}\, \delta(\vec{Y}_2 - \vec{Y}_1 - \vec{Y}_3) J_{1,l}(\alpha_1, \vec{x}_{1,l}, \vec{y}_{1,l}) J_{2,l}(\alpha_2, \vec{x}_{2,l}, \vec{y}_{2,l}) \\ &\cdot f_{\{\vec{n}_{3,l}\}}(\vec{y}_{3,l}) e^{-i\tau p_3^-} e^{-ip_3^+ x_2^-} e^{-ip_1^+ \Delta x^-} . \end{aligned} \tag{5.23}$$

Here we have introduced the shorthand notation

$$\begin{aligned} J_{r,l}(\alpha_r, \vec{x}_{r,l}, \vec{y}_{r,l}) \equiv& \left(\frac{\omega_{r,l}/2\alpha'}{\pi} \frac{1}{1 - e^{-i\tau_r \omega_{r,l}/|\alpha_r|}} \right)^{(d-2)/2} \\ &\cdot \exp \left\{ \frac{\omega_{r,l}/2\alpha'}{2i \sin\left(\frac{\tau_r \cdot \omega_{r,l}}{|\alpha_r|}\right)} \left[2\vec{x}_{r,l} \cdot \vec{y}_{r,l} - \left(\vec{x}_{r,l}^2 + \vec{y}_{r,l}^2\right) \cos\left(\frac{\tau_r \omega_{r,l}}{|\alpha_r|}\right) \right] \right\} \end{aligned} \tag{5.24}$$

and as before $\Delta x^- \equiv x_2^- - x_1^-$. The τ-dependence enters through $\tau_r = \tau - x_r^+$. For the kinematic situation we are considering here, the delta-functional is

$$\delta(\vec{Y}_2 - \vec{Y}_3 - \vec{Y}_1) = \prod_{m=0}^{\infty} \delta\left(\vec{y}_{2,m} - \sum_{n=0}^{\infty} \left(\left|\frac{\alpha_3}{\alpha_2}\right| X_{mn}^{(3)} \vec{y}_{3,n} + \left|\frac{\alpha_1}{\alpha_2}\right| X_{mn}^{(1)} \vec{y}_{1,n} \right) \right) . \tag{5.25}$$

The matrices X are for $r = 1, 3$

$$X_{mn}^{(r)} = \begin{cases} \tilde{X}_{mn}^{(r)} , & m > 0 , n > 0 \\ \frac{1}{\sqrt{2}} \tilde{X}_{m0}^{(r)} , & m > 0 \\ 1 , & m = 0 = n, \end{cases} \tag{5.26}$$

where for $m > 0, n \geq 0$,

$$\tilde{X}_{mn}^{(1)} = (-1)^n \frac{2m\beta}{\pi} \frac{\sin m\pi\beta}{m^2\beta^2 - n^2} , \quad \tilde{X}_{mn}^{(3)} = \frac{2m(\beta+1)}{\pi} \frac{\sin m\pi\beta}{m^2(\beta+1)^2 - n^2} , \tag{5.27}$$

and $\beta = \alpha_1/\alpha_2$, $\beta + 1 = -\alpha_3/\alpha_2$. The matrix is $X_{mn}^{(2)} = \delta_{mn}$. Also for our case $F(\alpha_1, \alpha_2) = 1$.

To proceed further, one may write $M^{(1)}$ in the form

$$M^{(1)} = \int_{-\infty}^{\infty} d\alpha_1 K(\alpha_1)\, e^{-i\alpha_1 \Delta x^-/\alpha'}, \tag{5.28}$$

where

$$\begin{aligned} K(\alpha_1) &= \frac{ig}{2\pi}\int_{x_1^+}^{x_2^+} d\tau \int \prod_{r=1}^{3}\prod_{l=0}^{\infty} d\vec{y}_{r,l}\delta(\vec{Y}_2 - \vec{Y}_1 - \vec{Y}_3) \\ &\cdot \prod_{l=0}^{\infty} J_{1,l}(\alpha_1, \vec{x}_{1,l}, \vec{y}_{1,l}) J_{2,l}(\alpha_2, \vec{x}_{2,l}, \vec{y}_{2,l}) \cdot f_{\{\vec{n}_{3,l}\}}(\vec{y}_{3,l}) e^{-i\tau p_3^-} e^{-ip_3^+ x_2^-}. \end{aligned} \tag{5.29}$$

Now let us focus our attention on the α_1 integral. As was done in [58], [1], we can write $M^{(1)}$ as

$$\int_0^{\infty} d\alpha_1 K(\alpha_1) e^{-i\alpha_1 \Delta x^-/\alpha'} + \int_0^{\infty} d\alpha_1 K(-\alpha_1) e^{i\alpha_1 \Delta x^-/\alpha'}. \tag{5.30}$$

Rotate the first integral by sending $\alpha_1 \to i\alpha_1$ and the second term by sending $\alpha_1 \to -i\alpha_1$. Then

$$M^{(1)} = i\int_0^{\infty} d\alpha_1 K(i\alpha_1) e^{\alpha_1 \Delta x^-/\alpha'} - i\int_0^{\infty} d\alpha_1 K(i\alpha_1) e^{\alpha_1 \Delta x^-/\alpha'}. \tag{5.31}$$

If each individual integral converges, the two terms cancel each other and hence $M^{(1)} = 0$. For that, we must examine the large α_1 behavior of $K(i\alpha_1)$. It was at this point that the calculation for the flat background failed to vanish.

The above analysis was carried out for the general case with arbitrary string fields. It will be illuminating to consider a simplified situation where the 1$^{\text{st}}$ and 2$^{\text{nd}}$ string fields are taken to be the lowest component fields with:

$$\vec{n}_{1,l} = \vec{n}_{2,l} = \begin{cases} 0, & \text{when } l \geq 1, \\ \text{arbitrary}, & \text{when } l = 0. \end{cases} \tag{5.32}$$

The component field is obtained by integrating the string field with $\prod_{l=1}^{\infty} d\vec{x}_l \varphi^l_{\{\vec{n}_l\}}(\vec{x}_l)$. This gives

$$T(\tau, x^-, \vec{x}) = \int \frac{dp^+}{2\pi} \sum_{\vec{n}_0} a(p^+, \vec{n}_0) e^{-i\left(x^+p^- + x^-p^+\right)} \varphi^0_{\{\vec{n}_0\}}(\vec{x}) + h.c.\,, \tag{5.33}$$

where we have defined $a(p^+, \vec{n}_0) \equiv A\left(p^+, \vec{n}_0, \{\vec{n}_{l\geq 1} = 0\}\right)$ and in the following we often denote the zero mode $\vec{x}_0$ by $\vec{x}$ for simplicity. Furthermore, we restrict the 3$^{\text{rd}}$ string to be the following spectator state:

$$|3\rangle = A(p_3^+, \{\vec{n}_{3,l}\})|0\rangle, \quad \text{with} \quad \vec{n}_{3,l} = 0, \text{ for all } l. \tag{5.34}$$

We note that $p_3^- = 0$.

Following the same procedures as above, it is easy to obtain (5.28) with $K(\alpha_1)$ now taking the form

$$K(\alpha_1) = \frac{ig\, e^{-ip_3^+ x_2^-}}{2\pi} \int_{x_1^+}^{x_2^+} d\tau \int \prod_{r=1}^{3} \prod_{l=0}^{\infty} d\vec{y}_{r,l} \delta(\vec{Y}_2 - \vec{Y}_1 - \vec{Y}_3)$$

$$\cdot J_{1,0}(\alpha_1, \vec{x}_{1,0}, \vec{y}_{1,0}) J_{2,0}(\alpha_2, \vec{x}_{2,0}, \vec{y}_{2,0}) \prod_{l=1}^{\infty} \varphi^l_{\{0\}}(\vec{y}_{1,l}) \varphi^l_{\{0\}}(\vec{y}_{2,l}) \cdot f_{\{0\}}(\vec{y}_{3,l}). \quad (5.35)$$

We note that, compared with (5.29), the product $\prod_{l=1}^{\infty} J_{1,l}(\cdot) J_{2,l}(\cdot)$ in the second line there is replaced by $\prod_{l=1}^{\infty} \varphi^l_{\{0\}}(\vec{y}_{1,l}) \varphi^l_{\{0\}}(\vec{y}_{2,l})$ above due to the condition (5.32). Now we perform the contour rotation and focus on the integrals of the $\vec{y}$'s. Let us first integrate $d\vec{y}_{2,l}, l \geq 1$ using the nonzero mode delta functions. The resulting integral of $\vec{y}_{1,l}$ and $\vec{y}_{3,l}$, $l \geq 1$ is independent of the zero modes $\vec{y}_{1,0}$ and $\vec{y}_{3,0}$ in the large α_1 limit and so can be calculated easily. Next we integrate out $d\vec{y}_{3,0}$ using the zero mode delta function. Therefore in the large α_1 limit,

$$e^{ip_3^+ x_2^-} \frac{K(i\alpha_1)}{ig} \sim \int_{x_1^+}^{x_2^+} d\tau \int d\vec{y}_1 d\vec{y}_2 \; \hat{J}_{1,0} \hat{J}_{2,0} \exp\left[+ \frac{\omega_{3,0}}{4\alpha'} \left(\left|\frac{\alpha_2}{\alpha_3}\right| \vec{y}_2 - \left|\frac{\alpha_1}{\alpha_3}\right| \vec{y}_1 \right)^2 \right] \quad (5.36)$$

up to an unimportant α_1-dependent proportional factor which is sub-dominant in large α_1 limit. Here we have denoted $\vec{y}_{r,0}$ by $\vec{y}_r$ for simplicity. Also we have used the hat $\hat{}$ to denote the corresponding quantities with the substitution $\alpha_1 \to i\alpha_1$. For example, $\hat{\omega}_{1,l} = \sqrt{l^2 - \mu^2 \alpha_1^2}$ in $\hat{J}_{1,0}$. After the contour rotation and taking the large α_1 limit, we have

$$\hat{\alpha}_2 \sim -i\alpha_1, \quad \text{and} \quad \hat{\omega}_{1,l},\, \hat{\omega}_{2,l} \sim i\mu\alpha_1. \quad (5.37)$$

Now $\hat{J}_{r,0}$ takes the form

$$\hat{J}_{r,0} \sim \exp\left(-A_r \left(\vec{x}_r^2 + \vec{y}_r^2\right) + 2B_r \vec{x}_r \cdot \vec{y}_r\right) \quad (5.38)$$

with

$$A_r = \frac{\mu\alpha_1/(2\alpha')}{2\tan(\mu|\tau_r|)}, \quad B_r = \frac{\mu\alpha_1/(2\alpha')}{2\sin(\mu|\tau_r|)} \quad (5.39)$$

and thus the $\vec{y}_1, \vec{y}_2$ integral takes the form $\int d\vec{y}_1 d\vec{y}_2 \exp(-\sum_{r,s} N^{rs} \vec{y}_r \cdot \vec{y}_s + \sum_r \vec{S}_r \cdot \vec{y}_r)$ and can be easily carried out. We obtain for the $\vec{y}_1, \vec{y}_2$ integral in (5.36),

$$\exp\left[-\frac{\mu\alpha_1/(2\alpha')}{2\tan(\mu\Delta x^+)} (\vec{x}_1^2 + \vec{x}_2^2) + \frac{\mu\alpha_1/(2\alpha')}{\sin(\mu\Delta x^+)} \vec{x}_1 \cdot \vec{x}_2 \right] \quad (5.40)$$

in the leading large α_1 limit. It is remarkable that the various coefficients of N^{rs}, $\vec{S}_r$ combine to make the result (5.40) τ independent. Hence the τ integral in (5.36) can be calculated trivially. Finally we obtain for (5.31)

$$\int_0^\infty d\alpha_1 K(i\alpha_1) e^{\frac{\alpha_1}{\alpha'}\Delta x^-}$$
$$\sim \int_0^\infty d\alpha_1 \exp\left[\frac{\alpha_1}{\alpha'}\Big(\Delta x^- - \frac{\mu}{2\sin\left(\mu\Delta x^+\right)}\big((\vec{x}_1^2+\vec{x}_2^2)\cos\left(\mu\Delta x^+\right) - 2\vec{x}_1\cdot\vec{x}_2\big)\Big)\right] \quad (5.41)$$

The exponent of the integrand is precisely the tree level string light cone (4.56) restricted to the zero modes. Thus we have shown that, unlike the flat case, the matrix element (5.6) does not receive contribution from region outside the free string light cone.

To be fully supersymmetric, the bosonic vertex has to be completed with the fermionic vertex and a prefactor which is needed for the preservation of the supersymmetries. Also the bosonic string field has to be replaced by the light cone string superfield [45,46] so that we commute the matrix element of the commutator of two string superfields. Now the Grassmannian factor makes sub-dominant contributions to the contour-deformed integral in the limit $p^+ \to \infty$ and does not affect the convergence of the contour-deformed integral. This is also the case for the prefactor as it is a polynomial in p^+. Therefore we conclude that the commutator of the string fields is unaffected by the plane wave string interaction.

This result is surprising. In the flat background case, the string field commutator was found to receive additional non vanishing contribution ([61], [62] and above) from the interaction even if the two strings were outside the free theory string light cone. Since the plane wave background and the bosonic part of the vertex are both continuous in the $\mu \to 0$ limit, one may naively thought that the plane wave string field commutator should also receive additional contributions, at least in a neighborhood of $\mu = 0$. Our result shows that this is not the case and the matrix element (5.6) is discontinuous at $\mu = 0$. Technically the reason for the discontinuity is because the $\mu \to 0$ limit does not commute with the procedure of summing up the contributions from the infinite tower of string states.

Since the plane wave light cone string field theory is not continuously connected with the flat space string theory, there is no compelling reason to require that the 3-string vertex to be continuous at $\mu = 0$. What about the Z_2 symmetry? Without additional input, one cannot fix the form of the light cone vertex in the plane wave background uniquely from the supersymmetries alone. Imposing the Z_2 symmetry is enough to fix the vertex. However there is a possibility that the symmetry is spontaneously broken. Since the plane wave background is obtained from the AdS background by performing a Penrose limit, a reasonable possibility that may help to understand better the plane wave string interaction is to try perform this limit carefully on the dynamics on the AdS side. This interesting idea has been pursued recently by Dobashi and Yoneya [95], with further work by Lee and Russo [96]. They propose that the plane wave string vertex that is relevant for the holographic plane wave/SYM correspondence is given by the equal weighted sum of the Z_2-invariant vertex and the μ-continuous vertex. It turns out that this particular combination coincides with the vertex proposed previously in [111]. These authors also provide some intuitive understanding of the role of each parts of the vertex: the Z_2-invariant vertex describes the "bare" interaction, while the μ-continuous vertex describes the mixing of the BMN operators. Thus according to this proposal, not only the continuity of μ is not maintained, also the Z_2 symmetry is broken due to the mixing effects. The breaking of the Z_2 symmetry has also been revealed in previous field theory calculations [112], [113]. In principle, one can fix the form of the light cone vertex by starting from the covariant Witten string field theory and performing the light cone gauge fixing. To confirm this breaking from a more fundamental point of view will be very exciting.

5.3 Summary

In this chapter we investigated whether and how string interactions affect the string light cone. In point particle quantum field theories, this is not the case, as long as the interactions are local. The requirement of locality (and Lorentz covariance of course) for the interacting terms ensures that the theory respects the causal structure.

The case for the strings, as we saw is more interesting. By studying certain amplitudes, we saw that the string light cone in the flat background gets modified by interactions. This might be a gauge fixing artifact, since we performed the analysis using a gauge fixed theory, might be purely because the string is not local (as an extended object) or something more fundamental.

On the other hand, the exactly same analysis for the string in the plane wave background suggests that there is no modification due to the interactions. This is quite interesting and shows that the theory is not continuously connected to the flat background one. It also sheds some light into the choice of the 3-string vertex.

Chapter 6

Conclusions

In this final chapter, we conclude the thesis. After a brief summary of what we have discussed so far, we comment further on the string light cone and the impact that string interactions have on it. Finally we present possible extensions.

6.1 In summary

To summarise what we have done so far. We asked the question of causality in classical physics and we saw that the upper bound for the speed of any physical object places severe restrictions on the causal structure of any theory. This is summarised in the notion of the light cone. Then, we saw that in quantum theory (in quantum field theory to be precise), the demand of causality translates into the local commutativity of the field. To investigate causality in string theory, we mimicked quantum field theory. Specifically we formulated the condition for the string light cone to be the vanishing of the commutator of the string field.

String field theory, in contrast to point particle field theory is not a fully developed subject yet. There are many aspects of the theory that we do not fully understand and we do not know how to formulate them. Even though we lack a full covariant and gauge invariant string field theory, we saw that we can write one in the light cone gauge. The advantage of the light cone gauge is that we have a theory involving only the physical degrees of freedom, is manifestly unitary and trackable. As we saw in chapter 2, we can quantise the theory in a canonical formalism, reminding the Klein-Gordon field. This is one reason we restricted our analysis for the string in the light cone gauge.

The second reason is that we wanted to perform the same analysis in the plane wave background. As we saw in chapter 3, string theory in the plane wave background so far is trackable only in the light cone gauge. This is the second reason that we have to confine our analysis in the light cone gauge.

6.2 The String Light Cone

In order to study causality in string theory, we mimicked quantum field theory. We formulated the condition for causality as the vanishing of the commutator of two string fields, something that we described as microcausality. For strings propagating in a flat background, we found that the condition to determine whether two strings are causally related or not is (4.44) for open strings, with a similar expression for closed strings.

The first striking feature is that the string light cone does not match with the point particle light cone. At first glance this can be thought as alarming for the consistency of the theory, but we have to be careful, since strings are extended objects. Consider the case where we want to determine the causal relation of two strings and imagine that one of them is so small, that it shrinks to a point. Then, with respect to that string, we can define the light cone and it has the same shape as that of a point particle. The second string can be half inside, half outside this light cone and the string can still be causally related. Or it can be half inside, half outside and not causally related. By pure intuition, we can not resolve the situation. Equation (4.44) resolves this for us. Instead of considering it as a violation of causality, we should interpret it as what we mean by causality for extended objects. The naive application of the point particle light cone to strings (or any other extended object) does not apply. Are we going to use the light cone of the middle point, the end point, some other combination? And what about closed strings, where there is no way to single out a point? (4.44) and its counterpart for closed strings answer that question.

In support of that argument, notice that the modification of the light cone comes exclusively from the internal oscillating modes of the string. In the case that the string shrinks to a point, or equivalently one studies the centre of mass (with is described by the zero mode), one recovers the causality condition for point particles. If this was not the case, if the string light cone did modify the point particle light cone, then we should be worried.

In addition, recall that the first quantised theory of strings can be thought of as a two-dimensional quantum field theory[1] and its supersymmetric extension. This means that along the world sheet we have the causal structure of field theory as we know it and signals do not propagate faster than the speed of light. This fact adds further support to our claim that the string light cone does not really undermine the notion of light cone from point particles.

This characteristic carries over to the plane wave background analysis. Once more, the string light cone in the plane wave, is a generalisation of the point particle light cone (of the same background of course!), with the additional terms coming exclusively from the internal oscillating modes of the string.

Furthermore, notice that the string light cone in both cases respects the symmetries of the background. Minkowski space is translational invariant and this reflects to both the point particle light cone and the string light cone. On the other hand, for the plane wave translational invariance is lost, due to the extra μ term in the metric (3.7). Accordingly, the light cone exhibits the same lack of invariance.

Finally, notice that the light cone in the plane wave exhibits a periodicity in light cone time x^+, with period $2\pi/\mu$. This is a striking feature, since we have not imposed any periodicity condition. In formulating the theory, we started with the metric (3.7), without any further assumptions or comments about its origin. As far as we are concerned, we could just claim that we are smart enough (or lucky enough) to find that string theory is solvable in that particular background in the light cone gauge. Then, we formulate a string field theory and we calculate the string light cone. Lo and behold a periodicity appears! The only explanation for this is that the

[1]Actually it is a conformal field theory.

consistency of the theory requires such a periodicity. Of course, viewing the plane wave as the Penrose limit of $AdS_5 \times S^5$, this periodicity comes naturally into play.

6.3 Interactions

In quantum field theory, interactions that are local do not modify the causal structure of the theory. Actually, one goes the other way around. Causality should be respected from the full theory, not just the free part. The only way to ensure this is by adding the requirement of locality, especially for the interacting terms. This means that the fields (and their derivatives) that build up the interacting part of the Lagrangian of the theory, should all depend on the same point x. For example, for a real Klein-Gordon field, an interaction of the form $\phi^n(x)$ is permissible, an interaction of the form $\phi^n(x)\phi(y)$ is not.

Extended this to interaction terms in string theory is not straightforward. For one, locality is an ill-defined concept in a theory that can not be local because its fundamental object is extended in nature. The best we can do is to demand, that as the string breaks, the coordinates of the worldsheet change continuously. This is how we built up the interaction terms for string field theory.

Even so, we found out that for string in a flat background, interactions do not preserve the causal structure of the theory. Specifically, we saw via explicit calculation of certain amplitudes, that they receive contributions outside the string light cone. This can be really alarming.

Remember however, that we are working in a gauge fixed theory. Imposing the light cone gauge has rendered the theory manifestly unitary and free of ghosts, but this came at a price. String gauge symmetry has been lost completely and even worse, we do not know what it was originally. By string gauge symmetry we mean of course a transformation $\delta\Phi$ that will leave the action $S[\Phi, \ldots]$ of the full string field theory invariant. It could be then that this "violation of causality" is merely an artifact of the gauge fixing.

Think for a moment about QED. The theory is covariant and has a gauge symmetry. However, one can pass to the Coulomb gauge, where quantisation is straight forward and manifestly unitary. But there one discovers that he has action at a distance. This does not mean that the theory violates the laws of relativity, it is just an artifact of the gauge that we have to live with it. Similarly here, this might be a gauge artifact. But without the full theory, we can not tell for sure.

What perplexes things more is that the plane wave theory does not exhibit the same behavior. There we found that interactions respect the causal structure of the theory. Of course this calculation was carried out in a light cone (i.e. gauge fixed) string field theory. Which in return means that one gauge fixed version of the full string field theory results in an apparent causality violation, while another gauge fixed version does not. At this stage it would be impossible, and hence pointless, to carry arguing along that line, when we luck a full string field theory for the flat background, and things are at an even more primitive stage at the plane wave side.

This discontinuity between the flat and the plane wave theory has further consequences for the 3-string vertex in the plane wave background. Recall that our presentation in chapter 3 was for a vertex that respected the Z_2 symmetry explicitly. We argued that there could be a different vertex (as far as the fermionic part and the prefactor are concerned) by demanding that it rolls continuously to the flat vertex in the limit $\mu \to 0$. Given the background (metric plus the field strength) and the free theory, the continuity for $\mu \to 0$, sounds like a reasonable requirement. But not any more. The causal structure of the full theory in the plane wave background does not reduce to the flat case one. Therefore, there is no reason to demand smooth $\mu \to 0$ limit any more.

Is this result really in favour of the explicit Z_2 symmetry? It certainly looks like so. But there is a possibility that the symmetry is spontaneously broken. Since the plane wave background is obtained from the AdS background by performing a Penrose limit, a reasonable possibility, that may help to understand better the plane wave string interaction, is to try perform this limit carefully on the dynamics on the AdS side. This interesting idea has been pursued recently by Dobashi and Yoneya in [95], with additional work by Lee and Russo in [96]. They propose that the plane wave string vertex that is relevant for the holographic plane wave/SYM correspondence is given by the equal weighted sum of the Z_2-invariant vertex and the μ-continuous vertex. It turns out that this particular combination coincides with the vertex proposed previously in [111]. There, the authors also provide some intuitive understanding of the role of each parts of the vertex. The Z_2-invariant vertex describes the "bare" interaction, while the μ-continuous vertex describes the mixing of the BMN operators. Thus according to this proposal, not only the continuity of μ is not maintained, also the Z_2 symmetry is broken due to the mixing effects. The breaking of the Z_2 symmetry has also been revealed in previous field theory calculations [112], [113].

6.4 Extensions

How could we study string causality beyond the light cone gauge? The authors of [114] calculated the propagator of the string field in Siegel's string field theory[2], in a similar fashion as we did in the light cone gauge. Then they imposed the same causality condition as we did, that is they demanded the vanishing of the string field commutator. They found the same condition as in the light cone theory.

This is a good indication that the string light cone that has been obtained from light cone string field theory is actually the correct one, that can also be obtained from the full string field theory. Siegel's string field theory incorporates only open bosonic strings. Therefore, the result of [114] is in the right direction, but we feel that it would be interesting to get the string light cone from Witten's string field theory.

This is not any easy task as it might sound. Witten's string field theory has been quantised using the Batalin-Vilkovisky method [56], [57], see for example [55], but we feel that this is still incomplete. Any string theory involving open strings must also include closed strings. Even if one refuses to include them, they nevertheless make their appearance at one-loop calculations. With out the equivalent of Witten's theory for closed strings, the picture is incomplete.

[2]Siegel's string field theory stands somewhere between the light cone string field theory and Witten's. It is manifestly covariant, but lacks a string gauge symmetry. See [115], [116], [117] for the original formulation and [118], [119] for reviews.

Furthermore, the more interesting question is to examine within Witten's theory if and how interactions modify the causal structure of the theory. This will answer the question whether the modification of the string light cone found in [61], [62] is indeed a gauge fixing artifact or of more fundamental nature.

6.5 String causality and the S-matrix

Causality can also be formulated in terms of the S-matrix, specifically in terms of certain analyticity properties. For point particles, in the framework of quantum field theory it turns out that this is equivalent to the local requirement of vanishing of the commutator. In fact, one can derive the former from the later and vice versa.

In string theory, things are still at a very primitive stage. One problem is that the connection between the string field and the string S-matrix is not as straightforward as in point particle theory. Even more, the S-matrix requires asymptotic states for its definition. Although this can be done in the Minkowski spacetime, problems arise in the plane wave background. The periodicity in time, $x^+ \sim x^+ + 2\pi/\mu$, makes the definition of asymptotic states at least problematic. Formulating causality it terms of the S-matrix in the plane wave background has pathologies. On the other hand, as we saw in previous chapters, there is no problem with formulating causality in terms of the local commutativity of the field.

This does not mean that S-matrix causality is less important. On the contrary, it has much to say for the theory. In a recent publication, [120], the authors used a canonical quantisation scheme for Witten's theory, reviving an earlier work [121] to show that the theory is local in time and furthermore, causal.

6.6 Future directions

The different shape of the string light cone, compared to the point particle one, can have direct consequences for black hole physics. The event horizon of a black hole is in direct connection with the causal structure of spacetime. If this structure is modified by passing from point particle to strings, it will be very interesting to see how the notion of the event horizon changes. In other words, an object that is a black hole in the point particle sense, is also a black hole in string sense? This is a question that ultimately will have to be answered, although we do not know how to address it with the current technology.

Connected to that is the information loss paradox for the black holes. The authors of [62] used the apparent violation of string causality in the flat background in order to propose a solution. But under the recent light, that there is no such violation for string theory in the plane wave background, this needs to be reexamined. Even more, before we determine whether this is a real violation, or a gauge fixing artifact, there is not much that we can say.

In addition, one might wonder how this carries on for objects of higher dimensionality. We know that string theory contains D(p)-branes and M-theory is speculated to be a theory of M2 and M5 branes. Take the case of a membrane for simplicity. The head-on attack would be to formulate a field theory of membranes. If we are lucky enough to be able to quantise it canonically, then we can have the propagator and examine when the commutator of two membrane fields vanishes identically. If only things were so simple! Membrane theories are far more complicated than string theories. For the bosonic membrane, there exists a gauge (also called light cone) where the Hamiltonian is quadratic and the equations of motion are simple wave equations, see [122]. The problem is that the gauge fixing conditions are so complicated that we have been unable to manipulate them the way we did for the string. Remember that for the string, in the light cone gauge the degrees of freedom were the transverse X^i, x^+ and x_0^-. We used the constraints of the theory to determine $X^-(\sigma)$ in terms of the transverse coordinates and the light cone time. Restoring the x_0^- dependance was a simple matter of Fourier transforming. For membranes, it turns out that we can make a gauge fixing that eliminates X^- and X^{D-2}, up to integrations constants. The restoration of x_0^- is again a simple matter of Fourier transformation. On the other hand, X^{D-2} is constrained by a highly non-linear second order differential equation, which we have been unable to solve.

6.7 Epilogue

At this point it would be wise to pause. Examining causality in string theory, as the local commutativity of the string field has revealed many interesting features of the theory. Nevertheless, this is not the end as open questions remain and other await. We only hope that with this work, we have contributed a small part to the understanding of string theory and, more generally, to the understanding of Nature.

Appendix A

String Theory in the Light Cone Gauge

In this appendix, we present light cone string theory. The intention is to fix the notation and provide a brief review of the first quantised string for the reader that does not wish to visit the literature. However, we strongly encourage the reader who does not feel familiar with the subject to consult the literature.

A.1 Open Bosonic Strings

Our starting point is the Polyakov action

$$S = -\frac{1}{4\pi\alpha'}\int d\tau \int_0^\pi \delta\sigma \sqrt{h} h^{\alpha\beta}(\sigma) g_{\mu\nu}(X) \partial_\alpha X^\mu \partial_\beta X^\nu. \tag{A.1.1}$$

For a string moving in a flat background,

$$g^{\mu\nu}(X) = \eta^{\mu\nu}. \tag{A.1.2}$$

We take the metric to be with signature $(-, +, \ldots, +)$. The symmetries of the action (in particular, worldsheet reparametrization invariance and Weyl rescaling) imply constraints.

First we choose the conformal gauge,

$$h^{\alpha\beta} = \eta^{\alpha\beta}. \tag{A.1.3}$$

Then we change into light cone coordinates,

$$X^{\pm} \equiv \frac{1}{\sqrt{2}}(X^0 \pm X^{D-1}), \tag{A.1.4}$$

with the rest denoted as X^i $(i = 1, 2, \ldots, D-2)$ or $\vec{X}$ and referred as the transverse coordinates. Finally we impose the light cone gauge

$$X^+ = 2\alpha' p^+ \tau. \tag{A.1.5}$$

This choice of gauge simplifies the action considerably at the end we are left with

$$S = \frac{1}{4\pi\alpha'} \int d \int_0^{\pi} d\sigma \left(\partial_\tau X^i \partial_\tau X^i - \partial_\sigma X^i \partial_\sigma^i \right). \tag{A.1.6}$$

The equations of motion resulting are simply wave equations,

$$\partial_\tau^2 X^i - \partial_\sigma^2 X^i = 0, \tag{A.1.7}$$

supplemented by the (Neumann) boundary conditions

$$\partial_\sigma X^i \Big|_{\sigma=0,\pi} = 0. \tag{A.1.8}$$

The constraints,

$$(\partial_\tau X \pm \partial_\sigma X)^2 = 0, \tag{A.1.9}$$

can be used to solve for X^- in terms of X^+ and X^i, up to an integration constant x_0^-.

The solution to the equation of motion (A.1.7) are simple to solve. The solution is

$$X^i(\tau, \sigma) = x_0^i + 2\alpha' p_0^i \tau + i\sqrt{2\alpha'} \sum_{n \neq 0} \frac{1}{n} \alpha_n^i e^{-in\tau} \cos n\sigma. \tag{A.1.10}$$

The canonical momentum of the string can be found from action (A.1.6) and it is

$$P^i = \frac{1}{2\pi\alpha'} \partial_\tau X^i. \tag{A.1.11}$$

The Hamiltonian is then

$$H = \pi\alpha' \int_0^{\pi} d\sigma \left((\vec{P}(\tau, \sigma))^2 + \frac{1}{2\pi\alpha'} (\partial_\sigma \vec{X}(\tau, \sigma))^2 \right). \tag{A.1.12}$$

Quantisation is achieved by imposing the canonical commutation relations

$$\left[X^i(\tau, \sigma), P^j(\tau, \sigma') \right] = i\delta^{ij} \delta(\sigma - \sigma'), \tag{A.1.13}$$

the rest vanishing. In terms of the modes, they read

$$[\alpha^i_m, \alpha^j_n] = m\delta^{ij}\delta_{m,-n}, \tag{A.1.14}$$

$$[x^i_0, p^j_0] = \delta^{ij}, \tag{A.1.15}$$

with the rest vanishing. For $n > 0$, α_n is an annihilation operator, while α_{-n} in a creation operator. Their role is to create/destroy oscillating modes in the string. To see this better, define the rescalled operators

$$a^i_n = \sqrt{n}\alpha^i_n, n > 0, \tag{A.1.16}$$

$$a^{i\dagger}_n = \sqrt{n}\alpha^i_{-n}, n > 0. \tag{A.1.17}$$

They satisfy the commutation relations

$$[a^i_n, a^{i\dagger}_n] = \delta^{ij}\delta_{nm}. \tag{A.1.18}$$

For $\tau = 0$, we can write the mode expansion of the coordinates as

$$X^i(\sigma) = x^i_0 + \sqrt{2}\sum_{n=1}^{\infty} x^i_n \cos n\sigma, \tag{A.1.19}$$

where we have defined

$$x^i_n \equiv i\sqrt{\frac{\alpha'}{n}}\left(a^i_n - a^{i\dagger}_n\right). \tag{A.1.20}$$

A.2 Closed Strings

The case of the bosonic closed string is similar. We take the range of σ to be $o \leq \sigma \leq 2\pi$. The gauge fixing condition is now

$$X^+ = \alpha' p^+ \tau. \tag{A.2.21}$$

The string is required to obey the same wave equation of motion (A.1.7), supplemented now by the periodicity condition

$$\sigma \sim \sigma + 2\pi, \tag{A.2.22}$$

suitable for the closed string.

The solution to the equation of motion reads now

$$X^i(\tau,\sigma) = x_0^i + \alpha' p_0^i \tau + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n} \left(\alpha_n^i e^{-in(\tau-\sigma)} + \tilde{\alpha}_n^i e^{-in(\tau+\sigma)}\right). \qquad (A.2.23)$$

Notice that the closed string has double the number of modes of the open string, except of the zero mode.

Quantising the theory is similar to the open string case. In terms of the modes, the canonical commutation relations read

$$[\alpha_m^i, \alpha_n^j] = m\delta^{ij}\delta_{m,-n}, \qquad (A.2.24)$$

$$[\tilde{\alpha}_m^i, \tilde{\alpha}_n^j] = m\delta^{ij}\delta_{m,-n}, \qquad (A.2.25)$$

$$[x_0^i, p_0^j] = \delta^{ij}, \qquad (A.2.26)$$

with the rest vanishing.

For $\tau = 0$, we can write the mode expansion for the coordinates as

$$X^i(\sigma) = x_0^i + \sqrt{2}\sum_{n=1}^{\infty} \left(x_n^i \cos n\sigma + \tilde{x}_n^i \sin n\sigma\right). \qquad (A.2.27)$$

To do that, first we define (suppressing spacetime indices)

$$\alpha_n^{\mathrm{I}} = \frac{1}{\sqrt{2}} \left(\alpha_n \tilde{\alpha}_n\right), \qquad (A.2.28)$$

$$\alpha_n^{\mathrm{II}} = \frac{1}{\sqrt{2}} \left(\alpha_n - \tilde{\alpha}_n\right). \qquad (A.2.29)$$

Then we rescale them,

$$\alpha_n^{\mathrm{I,\,II}} = \sqrt{n} a_n^{\mathrm{I,\,II}}, n > 0, \qquad (A.2.30)$$

$$\alpha_n^{\mathrm{I,\,II}} = \sqrt{n} a_n^{\mathrm{I,\,II}\dagger}, n > 0 \qquad (A.2.31)$$

and finally we define

$$x_n = i\frac{\alpha'}{2n} \left(a^{\mathrm{I}}{}_{-}n - a_n^{\mathrm{I}\dagger}\right), \qquad (A.2.32)$$

$$\tilde{x}_n = -\frac{\alpha'}{2n} \left(a_n^{\mathrm{II}} + a_n^{\mathrm{II}\dagger}\right). \qquad (A.2.33)$$

A.3 Superstrings

In analogy with the bosonic string, we consider a string propagating in a flat superspace, with bosonic coordinates x^μ, $\mu = 0, 1, \dots, 9$ and fermionic coordinates θ^A, $A = 1, 2$. The generalisation of the Polyakov action is

$$\begin{aligned} S &= -\frac{1}{4\pi\alpha'}\int d^2\sigma\Big\{\sqrt{h}h^{\alpha\beta}\Pi_\alpha\cdot\Pi_\beta - 2i\epsilon^{\alpha\beta}\partial_\alpha X^\mu\left(\bar\theta^1\Gamma_\mu\partial_\beta\theta^1 - \bar\theta^2\Gamma_\mu\partial_\beta\theta^2\right) \\ &+ 2\epsilon^{\alpha\beta}\bar\theta^1\Gamma^\mu\partial_\alpha\theta^1\bar\theta^2\Gamma_\mu\partial_\beta\theta^2\Big\}, \end{aligned} \tag{A.3.34}$$

where

$$\Pi^\mu_\alpha \equiv \partial_\alpha X^\mu - i\bar\theta^A\Gamma^\mu\partial_\alpha\theta^A. \tag{A.3.35}$$

In order for this action to posses global spacetime supersymmetry, the fermions are required to be Majorana-Weyl (for spacetime dimensionality $D = 10$).

Notice that the bosonic part is the same as before, so we will not discuss it further. The light cone gauge for the fermionic coordinates is fixed by imposing the condition

$$\Gamma^+\theta^{1,2} = 0. \tag{A.3.36}$$

Then, the action simplifies dramatically and we have ($a = 1, 2, \dots, 8$ is a spinor index)

$$S = -\frac{1}{4\pi\alpha'}\int d^2\sigma\left\{\partial_\alpha X^i\partial^\alpha X^i - 2\alpha' i\bar\Theta^a\rho^\alpha\partial_\alpha\Theta^a\right\}, \tag{A.3.37}$$

where we have defined the new spinors

$$S = \sqrt{p^+}\theta^1, \tag{A.3.38}$$

$$\tilde S = \sqrt{p^+}\theta^2, \tag{A.3.39}$$

and we have combined them into a two component Majorana spinor,

$$\Theta = \begin{pmatrix} S \\ \tilde S \end{pmatrix}. \tag{A.3.40}$$

The matrices ρ^α are

$$\rho^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \rho^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \tag{A.3.41}$$

The equations of motion that follow are

$$\left(\frac{\partial^2}{\partial\sigma^2} - \frac{\partial^2}{\partial\tau^2}\right) X^i = 0, \tag{A.3.42}$$

$$\left(\frac{\partial}{\partial\tau} + \frac{\partial}{\partial\sigma}\right) S^a = 0, \tag{A.3.43}$$

$$\left(\frac{\partial}{\partial\tau} - \frac{\partial}{\partial\sigma}\right) \tilde{S}^b = 0. \tag{A.3.44}$$

These are accompanied by suitable boundary conditions, periodicity in σ for closed strings and the requirement $S(0,\tau) = S(\pi,\tau)$, $\tilde{S}(0,\tau) = \tilde{S}(\pi,\tau)$ for open strings. Solving then the equation of motion results in

$$S^a = \frac{1}{\sqrt{2}} \sum_{n=-\infty}^{\infty} S_n^a e^{-in(\tau-\sigma)}, \tag{A.3.45}$$

$$\tilde{S}^a = \frac{1}{\sqrt{2}} \sum_{n=-\infty}^{\infty} S_n^a e^{-in(\tau+\sigma)}, \tag{A.3.46}$$

for open strings. Reality is imposed by the requirement $S^a_{-n} = (S^a_n)^\dagger$. For the closed strings we have

$$S^a = \sum_{n=-\infty}^{\infty} S_n^a e^{-in(\tau-\sigma)}, \tag{A.3.47}$$

$$\tilde{S}^a = \sum_{n=-\infty}^{\infty} \tilde{S}_n^a e^{-in(\tau+\sigma)}. \tag{A.3.48}$$

To quantise the theory, we impose the equal time anticommutation relations

$$\{S^a(\sigma,\tau), S^b(\sigma',\tau)\} = \pi\delta^{ab}\delta(\sigma-\sigma') = \{\tilde{S}^a(\sigma,\tau), \tilde{S}^b(\sigma',\tau)\}, \tag{A.3.49}$$

$$\{S^a(\sigma,\tau), \tilde{S}^b(\sigma',\tau)\} = 0. \tag{A.3.50}$$

In terms of the modes we have that

$$\{S_n^a, S_m^b\} = \delta^{ab}\delta_{n,-m} = \{\tilde{S}_n^a, \tilde{S}_m^b\}. \tag{A.3.51}$$

Evidently, S is both a coordinate and its conjugate momentum.

We need distinct coordinates and momenta for a Hamiltonian formulation. In general, this can not be done unless we break the $SO(8)$ transverse invariance. This is accomplished as follows, $SO(8) \to SO(6) \times SO(2) \sim SU(4) \times U(1)$. Then, we have for the first spinor

$$S^A(\sigma) + iS^{A+4}(\sigma) = \theta^A(s), \tag{A.3.52}$$

$$S^A(\sigma) - iS^{A+4}(\sigma) = \lambda_A(\sigma) \equiv \frac{\delta}{\delta\theta^A(\sigma)}. \tag{A.3.53}$$

Here $A = 1, 2, 3, 4$ labels the **4** of $SU(4)$. Similar relations hold for the second spinor $\tilde{S}$.

The quantisation conditions now read

$$\left\{\lambda_A(\sigma), \theta^B(\sigma')\right\} = \delta^A_B \delta(\sigma - \sigma'), \tag{A.3.54}$$

$$\left\{\theta^A(\sigma), \theta^A(\sigma')\right\} = 0 = \left\{\lambda_A(\sigma), \lambda_B(\sigma')\right\}, \tag{A.3.55}$$

with similar expressions for $\tilde{\theta}^A$, $\tilde{\lambda}_A$. The mode expansions for open strings at $\tau = 0$ are

$$\theta^A(\sigma) = \sum_{n=-\infty}^{\infty} \theta^A_n e^{in\sigma}, \quad \tilde{\theta}^A(\sigma) = \sum_{n=-\infty}^{\infty} \theta^A_m e^{-in\sigma}, \tag{A.3.56}$$

$$\lambda_A(\sigma) = \sum_{n=-\infty}^{\infty} \frac{1}{2\pi} \frac{\partial}{\partial \theta^A_{-n}} e^{in\sigma}, \quad \tilde{\lambda}_A(\sigma) = \sum_{n=-\infty}^{\infty} \frac{1}{2\pi} \frac{\partial}{\partial \theta^A_{-n}} e^{-in\sigma}. \tag{A.3.57}$$

For closed strings, they are

$$\theta^A(\sigma) = \sum_{n=-\infty}^{\infty} \theta^A_n e^{in\sigma}, \quad \tilde{\theta}^A(\sigma) = \sum_{n=-\infty}^{\infty} \tilde{\theta}^A_m e^{-in\sigma}, \tag{A.3.58}$$

$$\lambda_A(\sigma) = \sum_{n=-\infty}^{\infty} \frac{1}{2\pi} \frac{\partial}{\partial \theta^A_{-n}} e^{in\sigma}, \quad \tilde{\lambda}_A(\sigma) = \sum_{n=-\infty}^{\infty} \frac{1}{2\pi} \frac{\partial}{\partial \tilde{\theta}^A_{-n}} e^{-in\sigma}. \tag{A.3.59}$$

The fermionic part of the Hamiltonian that generates translations in τ can be obtained from (A.3.37)and it is

$$H_F = \pi\alpha' \int d\sigma \left\{ \frac{i}{2\pi^2\alpha'} \left(\theta' \frac{\delta}{\delta\theta} - \tilde{\theta}' \frac{\delta}{\delta\tilde{\theta}} \right) \right\}. \tag{A.3.60}$$

To study the interactions, it will be convenient to take the range of σ to be $0 \le \sigma \le \pi\alpha$ (for open strings, double that for closed). Then it is more convenient to rescale the modes for $\theta(\sigma), \tilde{\theta}(\sigma)$ as

$$\theta^A_m = \frac{1}{\sqrt{2}\alpha} R^A_m, \tag{A.3.61}$$

with a similar expression for $\tilde{\theta}^A_m$.

Appendix B

Implications of Microcausality in Quantum Field Theory

In this appendix we provide a very short presentation on how causality, formulated as the local commutativity of fields, affects the S-matrix. We also sketch the derivation of some interesting results. Our presentation is based on [9], but the interested reader might also want to consult [123].

B.1 S-matrix and causality

The S-matrix incorporates all the information one needs in order to find any scattering amplitude. Scattering experiments are done as follows. A collection of particles, that are quite far separated, are allowed to interact briefly in a small area and then they fly apart and far away. Ideally, this means that in the far past ($t \to -\infty$), the wavepackets that describe them are quite well localised in space and there is no overlap among them. Then, they are allowed to come in contact (overlap) and the interaction takes place. Finally, the particles that emerge from the interaction are allowed to fly away, so that in the far future ($t \to \infty$) their wavepackets are well localised in space without any overlap between them. Causality is fulfilled if first the incoming wavepackets reach the scattering region and then the outgoing wavepackets leave.

In perturbation theory, for a theory with the interaction term in the Hamiltonian being

$$V(t) = \int d^3x \mathcal{H}(x) \tag{B.1.1}$$

($\mathcal{H}$ being a Lorentz scalar), the S-matrix is

$$S = 1 + \sum_{n=1}^{\infty} \int d^4x_1 \cdots d^4x_n T\left\{\mathcal{H}(x_1) \cdots \mathcal{H}(x_n)\right\}. \tag{B.1.2}$$

The time ordering is a Lorentz invariant for spacelike separations. Therefore we have not introduced a special Lorentz frame if

$$[\mathcal{H}(x), \mathcal{H}(x')] = 0, \text{ for } (x - x')^2 \geq 0. \tag{B.1.3}$$

Clearly, for an interaction built up as a local function of fields (and their derivatives, which in a sense are other fields), equation (B.1.3) is automatically satisfied. For example, in the simple case of the real Klein-Gordon field, a local interaction of the form

$$\mathcal{H}(x) = (\phi(x))^n \tag{B.1.4}$$

fulfills this requirement.

B.2 Dispersion Relations

The requirement for causality, implies certain analytic behaviour for the S-matrix in point particle field theory. This is also known as the optical theorem.

Consider the scattering of a massless particle by an arbitrary target α, of mass $m_\alpha > 0$ and momentum $\mathbf{p}_\alpha = 0$. Let k be its initial momentum, k' its final. The S-matrix element is

$$\begin{aligned} S &= \frac{1}{(2\pi)^3\sqrt{4\omega\omega'}|N|^2} \lim_{\substack{k\to 0\\ k'\to 0}} \\ &\int d^4x \int d^4y e^{ik\cdot x - ik'\cdot y} \left(i\Box_x\right)\left(i\Box_y\right) \langle\alpha|T\left\{A^\dagger(y)A(x)\right\}|\alpha\rangle. \end{aligned} \tag{B.2.5}$$

Here it is $\omega \equiv k^0$, $\omega' \equiv k'^0$. The operators $A(x)$ can be any Heisenberg picture operators with non vanishing matrix element $\langle 0|A(x)|k\rangle = (2\pi)^{-3/2}(2\omega)^{-1/2}Ne^{ik\cdot x}$, between the vacuum and one-particle states. N is a constant. By letting the differential operators act on the As and denoting $\Box_x A(x) \equiv J(x)$, we have that

$$\begin{aligned} S &= \frac{-1}{(2\pi)^3\sqrt{4\omega\omega'}|N|^2} \lim_{\substack{k\to 0\\ k'\to 0}} \\ &\int d^4x \int d^4y e^{ik\cdot x - ik'\cdot y}\langle\alpha|T\left\{J^\dagger(y)J(x)\right\}|\alpha\rangle + \text{ETC}. \end{aligned} \tag{B.2.6}$$

ETC stands for terms that are the Fourier transforms of Equal Time Commutators, arising by the action of the differential operators acting on the step functions of the time ordered product. The equal time commutator of the operators $A(x)$, $A^\dagger(y)$ vanishes unless $\mathbf{x}-\mathbf{y}$ is zero. Thus, ETC are the Fourier transform of a differential operator acting on $\delta(x-y)$, with means that it is a polynomial of the momentum. For the analyticity properties of the S-matrix, its details are irrelevant.

Using the translation invariance, the matrix element is

$$S = -2\pi i\delta(k'-k)M(\omega), \tag{B.2.7}$$

where

$$M(\omega) = \frac{-i}{2\omega|N|^2}F(\omega), \tag{B.2.8}$$

and

$$F(\omega) = \int d^4x e^{i\omega q\cdot x}\langle\alpha|T\left\{J^\dagger(0)J(x)\right\}|\alpha\rangle + ETC, \tag{B.2.9}$$

where we have written k in terms of a fixed light like four-momentum q, $k^\mu = \omega q^\mu$, with $q^2 = 0$ and $q^0 = 1$.

The time ordered product can be written in two ways,

$$T\left\{J^\dagger(0)J(x)\right\} = \begin{cases} \vartheta(-x^0)\left[J^\dagger(0), J(x)\right] + J(x)J^\dagger(0), \\ -\vartheta(x^0)\left[J^\dagger(0), J(x)\right] + J^\dagger(0)J(x). \end{cases} \tag{B.2.10}$$

It follows that we can write for $F(\omega)$,

$$F(\omega) = F_A(\omega) + F_+(\omega) = F_R(\omega) + F_-(\omega), \tag{B.2.11}$$

where

$$F_A(\omega) \equiv \int d^4x\vartheta(-x^0)\langle\alpha|\left[J^\dagger(0), J(x)\right]|\alpha\rangle e^{i\omega q\cdot x} + \text{ETC}, \tag{B.2.12}$$

$$F_R(\omega) \equiv -\int d^4x\vartheta(x^0)\langle\alpha|\left[J^\dagger(0), J(x)\right]|\alpha\rangle e^{i\omega q\cdot x} + \text{ETC}, \tag{B.2.13}$$

$$F_+(\omega) \equiv \int d^4x\langle\alpha|J(x)J^\dagger(0)|\alpha\rangle e^{i\omega q\cdot x}, \tag{B.2.14}$$

$$F_-(\omega) \equiv \int d^4x\langle\alpha|J^\dagger(0)J(x)|\alpha\rangle e^{i\omega q\cdot x}. \tag{B.2.15}$$

Causality, formulated as the requirement for the local commutator of the fields to vanish, implies that the integrals (B.2.12) and (B.2.13) vanish, unless x^μ is inside the light cone. Then, the step function require that x^μ is in the past light cone for (B.2.12) so that $q \cdot x > 0$ and in the future light cone for (B.2.13), so that $q \cdot x < 0$. $F_A(\omega)$ is analytic for $\text{Im}(\omega) > 0$ and $F_R(\omega)$ is analytic for $\text{Im}(\omega) < 0$. We can define the function

$$\mathcal{F}(\omega) = \begin{cases} F_A(\omega), & \text{Im}(\omega) > 0, \\ F_R(\omega), & \text{Im}(\omega) < 0 \end{cases} \tag{B.2.16}$$

which is analytic in the whole ω-complex plane, except from a cut in the real axis.

The discontinuity of $\mathcal{F}$ along the real axis is

$$\mathcal{F}(E + i\epsilon) - \mathcal{F}(E - i\epsilon) = F_A(E) - F_R(E) = F_+(E) - F_-(E). \tag{B.2.17}$$

In the function $\mathcal{F}(\omega)/\omega^n$ vanishes for $\omega \to \infty$, by dividing with a polynomial $P(\omega)$ we obtain a function that also vanishes in the limit $\omega \to \infty$ and is analytic, except for the cut along the real axis and poles at the zeros ω_ν of the polynomial $P(\omega)$. By the calculus of residues, we have that

$$\frac{\mathcal{F}(\omega)}{P(\omega)} + \sum_\nu \frac{\mathcal{F}(\omega_\nu)}{(\omega_n - \omega)P'(\omega_\nu)} = \frac{1}{2\pi i}\oint_C \frac{\mathcal{F}(z)}{(z-\omega)}P(z)dz, \tag{B.2.18}$$

where ω is any point along the real axis and C is a contour consisting of two segments: one running just above the real axis, from $-\infty + i\epsilon$ to $\infty + i\epsilon$, and then around a large semicircle back to $-\infty + i\epsilon$ and a second running just below the real axis, from $\infty - i\epsilon$ to $-\infty - i\epsilon$ and then along a large semicircle back to $\infty - i\epsilon$. Thus we have

$$\mathcal{F}(\omega) = Q(\omega) + \frac{P(\omega)}{2\pi i}\int_{-\infty}^{\infty} dE \frac{F_-(E) - F_+(E)}{(E-\omega)P(E)}, \tag{B.2.19}$$

where Q is the $(n-1)$-order polynomial

$$Q(\omega) \equiv -P(\omega)\sum_\nu \frac{\mathcal{F}(\omega_\nu)}{(\omega_\nu - \omega)P'(\omega_n)}. \tag{B.2.20}$$

Equation (B.2.19) is the dispersion relation. The usefulness becomes more transparent, when one expresses the functions F_+, F_- in terms of measurable cross sections.

B.3 The Källen–Lehmann Representation

Here we present another application of microscopic causality in field theory.

Consider a complex scalar field in the Heisenberg picture. Consider the vacuum expectation value

$$\langle \phi(x)\phi^\dagger(y)\rangle. \tag{B.3.21}$$

It is

$$\langle \phi(x)\phi^\dagger(y)\rangle = \sum_n \langle 0|\phi(x)|n\rangle\langle n|\phi^\dagger(y)|0\rangle. \tag{B.3.22}$$

Choosing the states $|n\rangle$ to be momentum eigenstates, we have that

$$\langle 0|\phi(x)|n\rangle = e^{ip_n\cdot x}\langle 0|\phi(0)|n\rangle, \tag{B.3.23}$$

$$\langle n|\phi^\dagger(y)|0\rangle = e^{-ip_n\cdot x}\langle n|\phi^\dagger(0)|n\rangle \tag{B.3.24}$$

and thus

$$\langle \phi(x)\phi^\dagger(y)\rangle = \sum_n e^{-p_n\cdot(x-y)}|\langle 0|\phi(0)|n\rangle|^2. \tag{B.3.25}$$

Now, the sum

$$\sum_n \delta(p-p_n)|\langle 0|\phi(0)|n\rangle|^2 \tag{B.3.26}$$

is a scalar function of the momentum p^μ and therefore can depend only on p^2 and $\vartheta(p^0)$ for $p^2 \leq 0$. Actually, the intermediate states have $p^2 \leq 0$ and $p^0 > 0$, so this sum can be written as

$$\sum_n \delta(p-p_n)|\langle 0|\phi(0)|n\rangle|^2 = (2\pi)^{-3}\vartheta(p^0)\rho(-p^2), \tag{B.3.27}$$

with $\rho(-p^2) = 0$ for $p^2 > 0$. The function ρ is call *spectral function* and it is real and positive. Then, we have

$$\begin{aligned}\langle \phi(x)\phi^\dagger(y)\rangle &= (2\pi)^{-3}\int d^4p e^{ip\cdot(x-y)}\vartheta(p^0)\rho(-p^2) \\ &= (2\pi)^{-3}\int d^4p \int_0^\infty d\mu^2 e^{ip\cdot(x-y)}\vartheta(p^0)\rho(\mu^2)\delta(p^2+\mu^2).\end{aligned} \tag{B.3.28}$$

Interchanging the order of p and μ^2 integrations, we have that

$$\langle\phi(x)\phi^\dagger(y)\rangle = \int_0^\infty d\mu^2\rho(\mu^2)\Delta_+(x-y;\mu^2). \tag{B.3.29}$$

Δ_+ is the integral (4.16).

Similarly, we can show that

$$\langle\phi^\dagger(y)\phi(x)\rangle = \int_0^\infty d\mu^2\bar{\rho}(\mu^2)\Delta_+(y-x;\mu^2) \tag{B.3.30}$$

with $\bar{\rho}$ defined similarly to ρ,

$$\sum_n \delta(p-p_n)|\langle n|\phi^\dagger(0)|0\rangle|^2 = (2\pi)^{-3}\vartheta(p^0)\bar{\rho}(-p^2). \tag{B.3.31}$$

The requirement of causality demands that the commutator $[\phi(x), \phi^\dagger(y)]$ vanishes for spacelike separations. On the other hand, the commutator is

$$[\phi(x), \phi^\dagger(y)] = \int_0^\infty d\mu^2 \left(\rho(\mu^2)\Delta_+(x-y;\mu^2) - \bar{\rho}(\mu^2)\Delta_+(y-x;\mu^2)\right). \tag{B.3.32}$$

For spacelike separated x, y, the integral Δ_+ does not vanish but it is even. Thus, the only way for the commutator (B.3.32) to vanish is to have

$$\rho(\mu^2) = \bar{\rho}(\mu^2). \tag{B.3.33}$$

This is special case of the CPT theorem, proved here without resorting to perturbation theory.

Appendix C

Calculation of Amplitude $\mathcal{A}$

In this chapter we will calculate the amplitude (5.2), namely

$$\mathcal{A} = \langle 0|\Phi[p_3^+, \vec{X}_{(3)}(\sigma)] \left[\dot{\Phi}[x^-_{(1),0}, \vec{X}_{(1)}(\sigma)], \Phi[x^-_{(2),0}, \vec{X}_{(2)}(\sigma)]\right] |0\rangle, \tag{C.0.1}$$

in both the flat background and the plane wave background. We will show that it does not vanish outside the string light cone (for the flat background), while in the plane wave it is identically zero.

The dot stands for a time derivative and all fields are taken at the same time, namely $x^+ = 0$.

C.1 The flat background calculation

Let us present first the calculation for the flat background[1]. For $\dot{\Phi}$, we can use the Heisenberg equation of motion,

$$\dot{\Phi} = i\,[H, \Phi]\,. \tag{C.1.2}$$

Since we are interesting in potential deviations from the string light cone of the free theory, it will be sufficient to calculate

$$\mathcal{A} = i\langle 0|\Phi[p_3^+, \vec{X}_{(3)}(\sigma)] \left[\left[H_3, \Phi[x^-_{(1),0}, \vec{X}_{(1)}(\sigma)]\right], \Phi[x^-_{(2),0}, \vec{X}_{(2)}(\sigma)]\right] |0\rangle, \tag{C.1.3}$$

where H_3 is of course the 3-string interaction (2.35).

[1]This section is based on [61].

After some lengthy algebra, we have that

$$\begin{aligned}
\mathcal{A} &= ig \int \prod_{r=1}^{3} \left(d\beta_r \mathcal{D}\vec{Y}_{(r)}(\sigma) \right) \int dp_1^+ dp_2^+ \Big[\Theta(2,3)\Theta(3,2)\Theta(1,1) \\
&+ \Theta(1,1)\Theta(2,2)\Theta(3,3) + \Theta(1,3)\Theta(2,1)\Theta(3,2) \\
&+ \Theta(1,2)\Theta(2,1)\Theta(3,3) + \Theta(1,3)\Theta(2,2)\Theta(3,1) \\
&+ \Theta(1,2)\Theta(2,3)\Theta(3,1) \Big], \qquad \text{(C.1.4)}
\end{aligned}$$

where we have defined

$$\Theta(r,s) \equiv \delta\left[\vec{X}(r)(\sigma) - \vec{Y}_{(s)}(\sigma) \right] \delta(p_r^+ - \beta_{(s)}/2). \qquad \text{(C.1.5)}$$

Performing the integrals over β_r and $\vec{Y}_{(r)}$ and defining the vertex to be

$$\begin{aligned}
V(2p_1^+, \vec{X}_{(1)}; 2p_2^+, \vec{X}_{(2)}; 2p_3^+, \vec{X}_{(3)}) &\equiv \delta(p_1^+ + p_2^+ + p_3^+) \\
\mu(2p_1^+, 2p_2^+, 2p_3^+)\delta[\vec{X}_{(1)} - \vec{X}_{(2)} - \vec{X}_{(3)}], & \qquad \text{(C.1.6)}
\end{aligned}$$

we have for the amplitude that

$$\begin{aligned}
\mathcal{A} &= 2ig \int dp_1^+ dp_2^+ e^{-ix_1^- p_1^+ - ix_2^- p_2^+} \Big[V(2p_1^+, \vec{X}_{(1)}; 2p_2^+, \vec{X}_{(2)}; -2p_3^+, \vec{X}_{(3)}) \\
&+ V(2p_1^+, -\vec{X}_{(1)}; 2p_2^+, \vec{X}_{(2)}; -2p_3^+, -\vec{X}_{(3)}) \\
&+ V(2p_1^+, -\vec{X}_{(1)}; 2p_2^+, -\vec{X}_{(2)}; -2p_3^+, \vec{X}_{(3)}) \Big]. \qquad \text{(C.1.7)}
\end{aligned}$$

To make things more transparent, it will be sufficient to restrict ourselves to the zero modes of the vertex. It is then

$$\begin{aligned}
V(2p_1^+, \vec{X}_{(1)}; 2p_2^+, \vec{X}_{(2)}; -2p_3^+, \vec{X}_{(3)}) &= \delta\left(\sum_{r=1}^{3} \alpha_r \right) \delta\left(\sum_{r=1}^{3} \alpha_r \vec{x}_{(r),0} \right) \left(\frac{\alpha_1\alpha_2\alpha_3}{8\pi^3\tau_0} \right)^{(D-2)/2} \\
& \exp\left[\frac{\tau_0 m_0^2}{2} \sum_{r=1}^{3} \frac{1}{\alpha_r} + \frac{1}{2\tau_0} \sum_{r=1}^{3} \alpha_r \vec{x}_{(r),0}^2 \right]. \qquad \text{(C.1.8)}
\end{aligned}$$

With out loss of generality, we may assume that $\vec{X}_{(3)}$ is between $\vec{X}_{(1)}$ and $\vec{X}_{(2)}$, in which case only the first term in (C.1.7) survives (the other terms correspond to the two other kinematical situations). Then, the amplitude is

$$\begin{aligned}
\mathcal{A} &\propto \int_{-\infty}^{\infty} d\alpha_1 d\alpha_2 e^{-i(x_1^-\alpha_1 + x_2^-\alpha_2)/2} \left(\frac{\alpha_1\alpha_2\alpha_3}{8\pi^3\tau_0} \right)^{(D-2)/2} \delta\left(\sum_{r=1}^{2} \alpha_r \right) \\
& \exp\left[\frac{\tau_0 m_0^2}{2} \sum_{r=1}^{3} \frac{1}{\alpha_r} + \frac{1}{2\tau_0} \sum_{r=1}^{3} \alpha_r \vec{x}_{(r),0}^2 \right] \delta\left(\sum_{r=1}^{3} \alpha_3 \vec{x}_{(r),0} \right). \qquad \text{(C.1.9)}
\end{aligned}$$

Notice that we have two delta functions, so we can perform both integrations and the result is clearly no zero.

C.2 The plane wave calculation

Let us now repeat the same calculation in the plane wave background[2]. Notice that until we reach (C.1.7), the calculation is independent of whether we are working in the flat or the plane wave background. Let us again restrict our analysis in the zero modes only, using this time the plane wave vertex. In the momentum representation, it is

$$V = \delta\left(\sum_{r=1}^{3}\alpha_r\right)\delta\left(\sum_{r=1}^{3}\vec{p}_{(r),0}\right)\exp\left[\frac{1}{2}\sum_{r,s=1}^{3}a^{\dagger}_{(r),0}N^{rs}_{00}a^{\dagger}_{(s),0}\right]. \tag{C.2.10}$$

For the Neumann matrices, we have that

$$N^{rs}_{00} = (1-4\mu\alpha K)\left(\delta^{rs}+\frac{\sqrt{\alpha_r\alpha_s}}{\alpha_3}\right), \quad r,s=1,2, \tag{C.2.11}$$

$$N^{r3}_{00} = -\sqrt{\left|\frac{\alpha_r}{\alpha_3}\right|}, \tag{C.2.12}$$

$$N^{33}_{00} = 0. \tag{C.2.13}$$

Then we can write the vertex as

$$V = \delta\left(\sum_{r=1}^{3}\alpha_r\right)\delta\left(\sum_{r=1}^{3}\vec{p}_{(r),0}\right)\exp\left[\sum_{r=1}^{3}\frac{1}{A_r}\vec{p}_{(r),0}\right], \tag{C.2.14}$$

where we have defined

$$A_{r=1,2} = \frac{\mu\alpha_r}{m_0^2}\frac{1}{1-4\mu\alpha K}, \tag{C.2.15}$$

$$A_3 = \frac{\mu|\alpha_3|}{m_0^2}\frac{1}{1+4\mu\alpha K}. \tag{C.2.16}$$

Fourier transforming to position space results to

$$\begin{aligned}\tilde{V} \;=\; & \delta\left(\sum_{r=1}^{3}\right)\frac{1}{(4\pi)^{3(D-2)/2}}\left(-A_1A_2A_3\right)^{(D-2)/2} \\ & \exp\left[\frac{1}{4(A_1+A_2+A_3)}\Big(A_1A_3(\vec{x}_{(1),0}-\vec{x}_{(3),0})^2\right. \\ & \left.+A_2A_3(\vec{x}_{(2),0}-\vec{x}_{(3),0})^2+A_1A_2(\vec{x}_{(1),0}-\vec{x}_{(2),0})^2\Big)\right].\end{aligned} \tag{C.2.17}$$

Considering again, without loss of generality, the kinematical situation where $\vec{X}_{(3)}$ is between $\vec{X}_{(1)}$ and $\vec{X}_{(2)}$, we have for the amplitude that

$$\mathcal{A} \propto e^{ix_2^-\alpha_3/2}\int_{-\infty}^{\infty}d\alpha_1 e^{i(x_2^- - x_1^-)\alpha_1/2}I(\alpha_1), \tag{C.2.18}$$

[2]This section is based on unpublished joined work by C-S. Chu and K. Kyritsis.

where we have performed the α_2 integration using the delta function and we have abbreviated

$$I(\alpha_1) = \left(-\frac{A_1A_2A_3}{A_1+A_2+A_3}\right)^{(D-2)/2} \exp\left[\frac{1}{4(A_1+A_2+A_3)}\left(A_1A_3(\vec{x}_{(1),0}-\vec{x}_{(3),0})^2 + A_2A_3(\vec{x}_{(2),0}-\vec{x}_{(3),0})^2 + A_1A_2(\vec{x}_{(1),0}-\vec{x}_{(2),0})^2\right)\right], \tag{C.2.19}$$

with the understanding that $\alpha_2 = -\alpha_3 - \alpha_1$.

Consider the integral

$$J \equiv \int_{-\infty}^{\infty} d\alpha_1 e^{i(x_2^- - x_1^-)\alpha_1/2} I(\alpha_1). \tag{C.2.20}$$

Break it down to two parts,

$$J = J_1 + J_2, \tag{C.2.21}$$

where

$$J_1 \equiv \int_0^{\infty} d\alpha_1 e^{i(x_2^- - x_1^-)\alpha_1/2} I(\alpha_1) \tag{C.2.22}$$

and

$$J_2 \equiv \int_{-\infty}^{0} d\alpha_1 e^{i(x_2^- - x_1^-)\alpha_1/2} I(\alpha_1). \tag{C.2.23}$$

Without loss of generality, we may assume that $x_2^- - x_1^- > 0$. Then we can apply the contour deformation argument. Rotate J_1 by sending $\alpha_1 \to i\alpha_1$ and J_2 by sending $\alpha_1 \to -i\alpha_1$. Then $J_1 = -J_2$ and $J = 0$. That in return implies that the amplitude $\mathcal{A}$ is zero, contrary to the flat case!

Of course the validity of the argument depends of the convergence properties of the integrals. It can be shown that they are both well behaving and the argument holds indeed. This is the first indication that unlike the flat background theory, in string field theory in the plane wave background amplitudes do not receive extra contribution outside the string light cone.

Bibliography

[1] Chong-Sun Chu and Konstantinos Kyritsis. The string light cone in the pp-wave background. *Phys. Lett.*, B566:240–247, 2003.

[2] Chong-Sun Chu and Konstantinos Kyritsis. Local commutativity and causality in interacting pp-wave string field theory. 2004.

[3] Wolfgang Rindler. Introduction to special relativity. Clarendon Press, UK, (1991), 171p.

[4] B. F. Schutz. A first course in general relativity. Cambridge, Uk: Univ. Pr. (1985) 376p.

[5] R. M. Wald. General relativity. Chicago, USA: Univ. Pr. (1984) 491p.

[6] Steven Weinberg. Gravitation and cosmology: Principles and applications of the general theory of relativity. Cambridge, Massachusetts, USA: John Wiley & Sons, Inc. (1972) 657p.

[7] Charles C. Misner, Kip S. Thorne, and John Archibald Wheeler. Gravitation. Freeman (1973) 1279p.

[8] Michael E. Peskin and D. V. Schroeder. An introduction to quantum field theory. Reading, USA: Addison-Wesley (1995) 842 p.

[9] Steven Weinberg. The quantum theory of fields. vol. 1: Foundations. Cambridge, UK: Univ. Pr. (1995) 609 p.

[10] Steven Weinberg. The quantum theory of fields. vol. 2: Modern applications. Cambridge, UK: Univ. Pr. (1996) 489 p.

[11] Steven Weinberg. The quantum theory of fields. vol. 3: Supersymmetry. Cambridge, UK: Univ. Pr. (2000) 419 p.

[12] Michael B. Green, J. H. Schwarz, and Edward Witten. Superstring theory. vol. 1: Introduction. Cambridge, Uk: Univ. Pr. (1987) 469 P. (Cambridge Monographs On Mathematical Physics).

[13] Michael B. Green, J. H. Schwarz, and Edward Witten. Superstring theory. vol. 2: Loop amplitudes, anomalies and phenomenology. Cambridge, Uk: Univ. Pr. (1987) 596 P. (Cambridge Monographs On Mathematical Physics).

[14] D. Lust and S. Theisen. Lectures on string theory. *Lect. Notes Phys.*, 346:1–346, 1989.

[15] Elias Kiritsis. Introduction to superstring theory. 1997.

[16] J. Polchinski. String theory. vol. 1: An introduction to the bosonic string. Cambridge, UK: Univ. Pr. (1998) 402 p.

[17] J. Polchinski. String theory. vol. 2: Superstring theory and beyond. Cambridge, UK: Univ. Pr. (1998) 531 p.

[18] M. Kaku. Introduction to superstrings and m-theory. New York, USA: Springer (1999) 587 p.

[19] M. Kaku. Strings, conformal fields, and m-theory. New York, USA: Springer (2000) 531 p.

[20] C. V. Johnson. D-branes. Cambridge, USA: Univ. Pr. (2003) 548 p.

[21] Burton Zwiebach. A first course in string theory. Cambridge University Press (2004), 558p.

[22] G. Veneziano. Construction of a crossing - symmetric, regge behaved amplitude for linearly rising trajectories. *Nuovo. Cim.*, A57:190–197, 1968.

[23] E. Corrigan and D. B. Fairlie. Off-shell states in dual resonance theory. *Nucl. Phys.*, B91:527, 1975.

[24] Tom Banks, W. Fischler, S. H. Shenker, and Leonard Susskind. M theory as a matrix model: A conjecture. *Phys. Rev.*, D55:5112–5128, 1997.

[25] N. Ishibashi, H. Kawai, Y. Kitazawa, and A. Tsuchiya. A large-n reduced model as superstring. *Nucl. Phys.*, B498:467–491, 1997.

[26] John H. Schwarz. Lectures on superstring and m theory dualities. *Nucl. Phys. Proc. Suppl.*, 55B:1–32, 1997.

[27] Juan M. Maldacena. The large n limit of superconformal field theories and supergravity. *Adv. Theor. Math. Phys.*, 2:231–252, 1998.

[28] Jens Lyng Petersen. Introduction to the maldacena conjecture on ads/cft. *Int. J. Mod. Phys.*, A14:3597–3672, 1999.

[29] Ofer Aharony, Steven S. Gubser, Juan M. Maldacena, Hirosi Ooguri, and Yaron Oz. Large n field theories, string theory and gravity. *Phys. Rept.*, 323:183–386, 2000.

[30] Igor R. Klebanov. Tasi lectures: Introduction to the ads/cft correspondence. 2000.

[31] Eric D'Hoker and Daniel Z. Freedman. Supersymmetric gauge theories and the ads/cft correspondence. 2002.

[32] Juan M. Maldacena. Tasi 2003 lectures on ads/cft. 2003.

[33] Gerard 't Hooft. Dimensional reduction in quantum gravity. 1993.

[34] Leonard Susskind. The world as a hologram. *J. Math. Phys.*, 36:6377–6396, 1995.

[35] Daniela Bigatti and Leonard Susskind. Tasi lectures on the holographic principle. 1999.

[36] Raphael Bousso. The holographic principle. *Rev. Mod. Phys.*, 74:825–874, 2002.

[37] Amanda W. Peet. Tasi lectures on black holes in string theory. 2000.

[38] S. Mandelstam. Interacting string picture of dual resonance models. *Nucl. Phys.*, B64:205–235, 1973.

[39] S. Mandelstam. Interacting string picture of the neveu-schwarz-ramond model. *Nucl. Phys.*, B69:77–106, 1974.

[40] Michio Kaku and K. Kikkawa. The field theory of relativistic strings. pt. 2: Loops and pomerons. *Phys. Rev.*, D10:1823–1843, 1974.

[41] Michio Kaku and K. Kikkawa. The field theory of relativistic strings, pt. 1: Trees. *Phys. Rev.*, D10:1110, 1974.

[42] E. Cremmer and Jean-Loup Gervais. Combining and splitting relativistic strings. *Nucl. Phys.*, B76:209, 1974.

[43] E. Cremmer and Jean-Loup Gervais. Infinite component field theory of interacting relativistic strings and dual theory. *Nucl. Phys.*, B90:410–460, 1975.

[44] J. F. L. Hopkinson, R. W. Tucker, and P. A. Collins. Quantum strings and the functional calculus. *Phys. Rev.*, D12:1653, 1975.

[45] Michael B. Green and John H. Schwarz. Superstring interactions. *Nucl. Phys.*, B218:43–88, 1983.

[46] Michael B. Green, John H. Schwarz, and Lars Brink. Superfield theory of type ii superstrings. *Nucl. Phys.*, B219:437–478, 1983.

[47] Michael B. Green and John H. Schwarz. The structure of superstring field theories. *Phys. Lett.*, B140:33, 1984.

[48] Michael B. Green and John H. Schwarz. Superstring field theory. *Nucl. Phys.*, B243:475–536, 1984.

[49] J. G. Taylor, P. C. Bressloff, and A. Restuccia. Finite superstrings. Singapore, Singapore: World Scientific (1992) 378 p.

[50] A. Restuccia and J. G. Taylor. Light cone gauge analysis of superstrings. *Phys. Rept.*, 174:283–407, 1989.

[51] Edward Witten. Noncommutative geometry and string field theory. *Nucl. Phys.*, B268:253, 1986.

[52] Nathan Berkovits. Super-poincare covariant quantization of the superstring. *JHEP*, 04:018, 2000.

[53] Nathan Berkovits. Covariant quantization of the superstring. *Int. J. Mod. Phys.*, A16:801–811, 2001.

[54] Nathan Berkovits. Ictp lectures on covariant quantization of the superstring. 2002.

[55] Charles B. Thorn. String field theory. *Phys. Rept.*, 175:1–101, 1989.

[56] I. A. Batalin and G. A. Vilkovisky. Quantization of gauge theories with linearly dependent generators. *Phys. Rev.*, D28:2567–2582, 1983.

[57] Joaquim Gomis, Jordi Paris, and Stuart Samuel. Antibracket, antifields and gauge theory quantization. *Phys. Rept.*, 259:1–145, 1995.

[58] Emil J. Martinec. The light cone in string theory. *Class. Quant. Grav.*, 10:L187–L192, 1993.

[59] Emil J. Martinec. Strings and causality. 1993.

[60] David A. Lowe. Causal properties of string field theory. *Phys. Lett.*, B326:223–230, 1994.

[61] David A. Lowe, Leonard Susskind, and John Uglum. Information spreading in interacting string field theory. *Phys. Lett.*, B327:226–233, 1994.

[62] David A. Lowe, Joseph Polchinski, Leonard Susskind, Larus Thorlacius, and John Uglum. Black hole complementarity versus locality. *Phys. Rev.*, D52:6997–7010, 1995.

[63] P. Goddard, J. Goldstone, C. Rebbi, and Charles B. Thorn. Quantum dynamics of a massless relativistic string. *Nucl. Phys.*, B56:109–135, 1973.

[64] David J. Gross and Vipul Periwal. Heterotic string light cone field theory. *Nucl. Phys.*, B287:1, 1987.

[65] A. Pankiewicz. Strings in plane wave backgrounds. *Fortsch. Phys.*, 51:1139–1203, 2003.

[66] Marcus Spradlin and Anastasia Volovich. Light-cone string field theory in a plane wave. 2003.

[67] Darius Sadri and Mohammad M. Sheikh-Jabbari. The plane-wave / super yang-mills duality. 2003.

[68] Matthias Blau, Jose Figueroa-O'Farrill, Christopher Hull, and George Papadopoulos. A new maximally supersymmetric background of iib superstring theory. *JHEP*, 01:047, 2002.

[69] Matthias Blau, Jose Figueroa-O'Farrill, Christopher Hull, and George Papadopoulos. Penrose limits and maximal supersymmetry. *Class. Quant. Grav.*, 19:L87–L95, 2002.

[70] Matthias Blau, Jose Figueroa-O'Farrill, and George Papadopoulos. Penrose limits, supergravity and brane dynamics. *Class. Quant. Grav.*, 19:4753, 2002.

[71] Iosif Bena and Radu Roiban. Supergravity pp-wave solutions with 28 and 24 supercharges. *Phys. Rev.*, D67:125014, 2003.

[72] Iosif Bena, Joseph Polchinski, and Radu Roiban. Hidden symmetries of the ads(5) x s**5 superstring. *Phys. Rev.*, D69:046002, 2004.

[73] R. R. Metsaev and A. A. Tseytlin. Type iib green-schwarz superstrings in ads(5) x s(5) from the supercoset approach. *J. Exp. Theor. Phys.*, 91:1098–1114, 2000.

[74] R. R. Metsaev. On manifest su(4) invariant superstring action in ads(5) x s(5). *Class. Quant. Grav.*, 18:1245–1260, 2001.

[75] R. R. Metsaev and A. A. Tseytlin. Superstring action in ads(5) x s(5): kappa-symmetry light cone gauge. *Phys. Rev.*, D63:046002, 2001.

[76] David Berenstein, Juan M. Maldacena, and Horatiu Nastase. Strings in flat space and pp waves from n = 4 super yang mills. *JHEP*, 04:013, 2002.

[77] David Berenstein, Edi Gava, Juan M. Maldacena, K. S. Narain, and Horatiu Nastase. Open strings on plane waves and their yang-mills duals. 2002.

[78] Jan Christoph Plefka. Lectures on the plane-wave string / gauge theory duality. *Fortsch. Phys.*, 52:264–301, 2004.

[79] R. R. Metsaev. Type iib green-schwarz superstring in plane wave ramond-ramond background. *Nucl. Phys.*, B625:70–96, 2002.

[80] R. R. Metsaev and A. A. Tseytlin. Exactly solvable model of superstring in plane wave ramond- ramond background. *Phys. Rev.*, D65:126004, 2002.

[81] Kostas Skenderis and Marika Taylor. Open strings in the plane wave background. ii: Superalgebras and spectra. *JHEP*, 07:006, 2003.

[82] Kostas Skenderis and Marika Taylor. Open strings in the plane wave background. i: Quantization and symmetries. *Nucl. Phys.*, B665:3–48, 2003.

[83] Jong-wook Kim, Bum-Hoon Lee, and Hyun Seok Yang. Superstrings and d-branes in plane wave. *Phys. Rev.*, D68:026004, 2003.

[84] Yang-Hui He, John H. Schwarz, Marcus Spradlin, and Anastasia Volovich. Explicit formulas for neumann coefficients in the plane- wave geometry. *Phys. Rev.*, D67:086005, 2003.

[85] John H. Schwarz. The three-string vertex for a plane-wave background. 2003.

[86] Chong-Sun Chu, Valentin V. Khoze, Michela Petrini, Rodolfo Russo, and Alessandro Tanzini. A note on string interaction on the pp-wave background. *Class. Quant. Grav.*, 21:1999–2009, 2004.

[87] Chong-Sun Chu, Michela Petrini, Rodolfo Russo, and Alessandro Tanzini. String interactions and discrete symmetries of the pp-wave background. *Class. Quant. Grav.*, 20:S457–S464, 2003.

[88] Paolo Di Vecchia, Jens Lyng Petersen, Michela Petrini, Rodolfo Russo, and Alessandro Tanzini. The 3-string vertex and the ads/cft duality in the pp-wave limit. *Class. Quant. Grav.*, 21:2221–2240, 2004.

[89] Marcus Spradlin and Anastasia Volovich. Superstring interactions in a pp-wave background. *Phys. Rev.*, D66:086004, 2002.

[90] Marcus Spradlin and Anastasia Volovich. Superstring interactions in a pp-wave background. ii. *JHEP*, 01:036, 2003.

[91] Ari Pankiewicz. More comments on superstring interactions in the pp-wave background. *JHEP*, 09:056, 2002.

[92] A. Pankiewicz and Jr. Stefanski, B. pp-wave light-cone superstring field theory. *Nucl. Phys.*, B657:79–106, 2003.

[93] Peter Lee, Sanefumi Moriyama, and Jong-won Park. Cubic interactions in pp-wave light cone string field theory. *Phys. Rev.*, D66:085021, 2002.

[94] Peter Lee, Sanefumi Moriyama, and Jong-won Park. A note on cubic interactions in pp-wave light cone string field theory. *Phys. Rev.*, D67:086001, 2003.

[95] Suguru Dobashi and Tamiaki Yoneya. Resolving the holography in the plane-wave limit of ads/cft correspondence. 2004.

[96] Sangmin Lee and Rodolfo Russo. Holographic cubic vertex in the pp-wave. 2004.

[97] Ari Pankiewicz. An alternative formulation of light-cone string field theory on the plane wave. *JHEP*, 06:047, 2003.

[98] James Lucietti, Sakura Schafer-Nameki, and Aninda Sinha. On the exact open-closed vertex in plane-wave light-cone string field theory. *Phys. Rev.*, D69:086005, 2004.

[99] James Lucietti. A note on the uniqueness of the neumann matrices in the plane-wave background. 2004.

[100] James Lucietti, Sakura Schafer-Nameki, and Aninda Sinha. On the plane-wave cubic vertex. *Phys. Rev.*, D70:026005, 2004.

[101] A. Pankiewicz and Jr. Stefanski, B. On the uniqueness of plane-wave string field theory. 2003.

[102] F. Mandl and G. Shaw. Quantum field theory. Chichester, Uk: Wiley (1984) 354 P. (A Wiley-interscience Publication).

[103] J. D. Bjorken and S. D. Drell. Relativistic quantum field theory. (german translation). Bibliograph.Inst./mannheim 1967, 409 P.(B.i.- hochschultaschenbuecher, Band 101).

[104] J. Dimock. Locality in free string field theory. *J. Math. Phys.*, 41:40–61, 2000.

[105] J. Dimock. Locality in free string field theory. ii. *Annales Henri Poincare*, 3:613–634, 2002.

[106] J. Dimock. Local string field theory. 2003.

[107] E. Kiritsis and B. Pioline. Strings in homogeneous gravitational waves and null holography. *JHEP*, 08:048, 2002.

[108] Donald Marolf and Simon F. Ross. Plane waves: To infinity and beyond! *Class. Quant. Grav.*, 19:6289–6302, 2002.

[109] Donald Marolf and Simon F. Ross. Plane waves and spacelike infinity. *Class. Quant. Grav.*, 20:4119–4133, 2003.

[110] David Berenstein and Horatiu Nastase. On lightcone string field theory from super yang-mills and holography. 2002.

[111] Chong-Sun Chu and Valentin V. Khoze. Correspondence between the 3-point bmn correlators and the 3-string vertex on the pp-wave. *JHEP*, 04:014, 2003.

[112] Chong-Sun Chu, Valentin V. Khoze, and Gabriele Travaglini. Bmn operators with vector impurities, z(2) symmetry and pp- waves. *JHEP*, 06:050, 2003.

[113] George Georgiou, Valentin V. Khoze, and Gabriele Travaglini. New tests of the pp-wave correspondence. *JHEP*, 10:049, 2003.

[114] Hiroyuki Hata and Hajime Oda. Causality in covariant string field theory. *Phys. Lett.*, B394:307–314, 1997.

[115] W. Siegel. Covariantly second quantized string. *Phys. Lett.*, B142:276, 1984.

[116] W. Siegel. Covariantly second quantized string. 2. *Phys. Lett.*, B149:157, 1984.

[117] W. Siegel. Covariantly second quantized string. 3. *Phys. Lett.*, B149:162, 1984.

[118] Hiroyuki Hata, Katsumi Itoh, Taichiro Kugo, Hiroshi Kunitomo, and Kaku Ogawa. Covariant string field theory. *Phys. Rev.*, D34:2360, 1986.

[119] Hiroyuki Hata, Katsumi Itoh, Taichiro Kugo, Hiroshi Kunitomo, and Kaku Ogawa. Covariant string field theory. 2. *Phys. Rev.*, D35:1318, 1987.

[120] Theodore G. Erler and David J. Gross. Locality, causality, and an initial value formulation for open string field theory. 2004.

[121] Masahiro Maeno. Canonical quantization of witten's string field theory using midpoint light cone time. *Phys. Rev.*, D43:4006–4014, 1991.

[122] Itzhak Bars. Membrane symmetries and anomalies. *Nucl. Phys.*, B343:398–417, 1990.

[123] C. Itzykson and J. B. Zuber. Quantum field theory. New York, USA: Mcgraw-hill (1980) 705 P.(International Series In Pure and Applied Physics).